RESEARCH ARTICLE

The Completion of Factorial Vector of length 4

Selby Jose

Department of Mathematics, Institute of Science, Madam Cama Road, Fort, Mumbai 400-032.

E mail: selbyjose@gmail.com

Manuscript Details

Received : 10.07.2015
Accepted: 21.08.2015
Online Published: 30.08.2015

ISSN: 2322-0015

Editor: Dr. Chavhan Arvind

Cite this article as:
Selby Jose. The completion of Factorial Vector of length 4. *Int. Res. J. of Science & Engineering*, 2015; Vol. 3 (4):152-155.

ABSTRACT

In this paper, we compute the complection of the unimodular row (a_0, a_1, a_2^2, a_3^3) if (a_0, a_1, a_2, a_3) is unimodular.

Keywords: Unimodular rows, completion of vector.
Mathematics Subject Classification 2010: 11E57, 13C10, 15A63

1. INTRODUCTION

Let R be a commutative ring with 1. For any unimodular row $v = (a_0, \dots, a_r) \in R^{r+1}$ of length $r+1$,
one has the following surjective map.

$$\begin{array}{ccc} R^{r+1} & \xrightarrow{v} & R \\ e_i & \mapsto & a_{i-1} \end{array}$$

Let P_v denote its kernel. Then one has a split exactsequence:

$$0 \to P_v \to R^{r+1} \xrightarrow{v} R \to 0$$

Thus P_v is a projective module of rank *r*, which is 1-stablyfree, i.e. $P_v \oplus R \simeq R^{r+1}$.$P_v$is free if and only if *v* can be completed to an invertible matrix, i.e. *v* is*completable*.

In [5], R.G. Swan and J. Towber proved that *If P is a projective R[X]-module of rank 2 and* $X^2R[X]^2 \subseteq P \subseteq R[X]^2$, *then* $P \simeq R[X]^2$ .As a consequence, they concluded that if $(a, b, c) \in Um_3(R)$,then (a^2, b, c) can be completed to an invertible matrix.This result was explained and generalized by A.A. Suslin in hisdoctoral thesis [2] in the mid-seventies.There he proves that *if* $(a_0, a_1, \dots, a_r) \in Um_{r+1}(R)$, *then the unimodular row* $(a_0, a_1, a_2^2, \dots, a_r^r)$ *can always be completed to an invertible matrix.*

2. Preliminaries

In this section we recall a few definitions, state some results and fix some notations which will be used throughout this paper.

Definition 2.1 *A row $v = (v_1, v_2, \dots, v_r) \in R^r$ is said to be* ***unimodular*** *(of length r) if there exists elements $w_1, w_2, \dots, w_r$ in R such that $v_1 w_1 + v_2 w_2 + \cdots + v_r w_r = 1$. $Um_r(R)$ will denote the set of all unimodular rows $v \in R^r$.*

Definition 2.2 *A row $v = (v_1, v_2, \dots, v_r)$ is said to be completable if there exist an invertible matrix φ such that $e_1 \varphi = v$.*

We now state some examples (from [1]) of completable rows. Consider the coordinate ring of the real n-sphere,

$$R_n = \frac{\mathbb{R}[t_0, t_1, \dots t_n]}{(t_0^2 + t_1^2 + \cdots + t_n^2 - 1)}$$

Let $a_0, a_1, \dots, a_n$ be the images of $t_0, t_1, \dots, t_n$ in R_n and let v be the unimodular row $(a_0, a_1, \dots, a_n) \in Um_{n+1}(R_n)$.

For $n = 1$, $(a_0, a_1) \in Um_2(R_1)$ is completable, and its completion is $\begin{pmatrix} a_0 & a_1 \\ a_1 & -a_0 \end{pmatrix}$. Also for $n = 3$, $(a_0, a_1, a_2, a_3) \in Um_4(R_3)$ is completable, and its completion is

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & -a_0 & a_3 & -a_2 \\ a_2 & -a_3 & -a_0 & a_1 \\ a_3 & a_2 & -a_1 & -a_0 \end{pmatrix}$$

3. Completion of $(\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2^2}, \mathbf{a_3^3})$

In this section, we give an explicit computation of the completion of the unimodularrow $(a_0, a_1, a_2^2, a_3^3) \in Um_4(R)$.

Let $(a_0, a_1, a_2, a_3) \in Um_4(R)$. Consider the matrix,

$$\beta_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_3' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & b_3' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ -a_3^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_3 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_3^2(2a_3^2 b_3^2 - 4a_3 b_3 + 3) & -(1 - a_3 b_3)^2 & -a_3(1 - a_3 b_3)^2 \\ 2a_3(1 - a_3 b_3)^2 & b_3(2 - a_3 b_3) & -(1 - a_3 b_3)^2 \\ (1 - a_3 b_3)^2 & 0 & b_3(2 - a_3 b_3) \end{pmatrix}$$
$$= \begin{pmatrix} a_3^2(2a_3^2 b_3^2 - 4a_3 b_3 + 3) & -(1 - a_3 b_3)^2 & -a_3(1 - a_3 b_3)^2 \\ 2a_3(1 - a_3 b_3)^2 & b_3' & -(1 - a_3 b_3)^2 \\ (1 - a_3 b_3)^2 & 0 & b_3' \end{pmatrix}$$

where $b_3' = b_3(2 - a_3 b_3)$. Consider,

$$\beta_2 = \begin{pmatrix} -2a_3^3 & a_3 & a_3^2 \\ -2a_3^3 & 1 & a_3 \\ -a_3 & 0 & 1 \end{pmatrix}$$

Thus one has,

$$a_3 I_3 \beta_1 \;+ (1 - a_3 b_3)^2 \beta_2 \;= \begin{pmatrix} a_3^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. $a_3 I_3 \beta_1 \;+ det(\gamma)\beta_2 \;= \begin{pmatrix} a_3^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $\gamma \;= \begin{pmatrix} a_0 & a_1 & a_2^2 \\ b_1^2 & -b_2 - b_0 b_1 & -a_0 + 2a_2 b_1 \\ b_2 - b_0 b_1 & b_0^2 & -a_1 - 2a_2 b_0 \end{pmatrix}$.

Take

$$\beta \;= \begin{pmatrix} \gamma & a_3 I_3 \\ -b_3' I_3 & adj(\gamma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta_1 \end{pmatrix} \begin{pmatrix} 1 & adj(\gamma)\beta_2 \\ 0 & 1 \end{pmatrix}$$

One can write the above matrix βin the form

$$\beta \;= \begin{pmatrix} \gamma & \begin{pmatrix} a_3^3 & 0 \\ 0 & I_2 \end{pmatrix} \\ -b_3' & -b_3' \, adj(\gamma)\beta_2 \;+ adj(\gamma)\beta_1 \end{pmatrix}$$

Let $K \;= -b_3' \, adj(\gamma)\beta_2 \;+ adj(\gamma)\beta_1$. Then

$K_{11} \;= \;\; 2a_1^2 a_3 \;+ a_2^2 b_2 \;- a_0 a_1 \;+ 3a_1 a_3^2 b_2 \;+ a_2^2 b_0 b_1 \;+ 3a_0 a_3^2 b_0^2 \;+ 2a_2^2 a_3 b_0^2$

$\quad + 2a_1 a_2 b_1 \;+ 3a_1 a_3^2 b_0 b_1 \;+ 6a_2 a_3^2 b_0 b_2 \;+ 4a_1 a_2 a_3 b_0$

$K_{12} \;= \;\; -a_0 b_0^2 \;- a_1 b_2 \;- a_1 b_0 b_1 \;- 2\, a_2 b_0 b_2$

$K_{13} \;= \;\; -a_2^2 b_0^2 \;- a_1^2 \;- a_0 a_3 b_0^2 \;- 2a_1 a_2 b_0 \;- a_1 a_3 b_2 \;- a_1 a_3 b_0 b_1 \;- 2a_2 a_3 b_0 b_2$

$K_{21} \;= \;\; a_2^2 b_1^2 \;+ a_0^2 \;- 3a_0 a_3^2 b_2 \;- 2a_2^2 a_3 b_2 \;+ 3a_1 a_3^2 b_1^2 \;- 2a_0 a_1 a_3 \;- 2a_0 a_2 b_1$

$\quad + 3a_0 a_3^2 b_0 b_1 \;+ 2a_2^2 a_3 b_0 b_1 \;+ 6a_2 a_3^2 b_1 b_2 \;- 4a_0 a_2 a_3 b_0$

$K_{22} \;= \;\; -a_1 b_1^2 \;+ a_0 b_2 \;- a_0 b_0 b_1 \;- 2\, a_2 b_1 b_2$

$K_{23} \;= \;\; a_2^2 b_2 \;+ a_0 a_1 \;- a_1 a_3 b_1^2 \;- a_2^2 b_0 b_1 \;+ 2a_0 a_2 b_0 \;+ a_0 a_3 b_2 \;- a_0 a_3 b_0 b_1 \;- 2a_2 a_3 b_1 b_2$

$K_{31} \;= -a_1 b_1^2 \;+ 3a_3^2 b_2^2 \;- a_0 b_2 \;- 2a_0 a_3 b_0^2 \;+ 2a_1 a_3 b_2 \;- a_0 b_0 b_1 \;- \;2a_1 a_3 b_0 b_1$

$K_{32} \;= -b_2^2$

$K_{33} \;= \; a_0 b_0^2 \;- a_3 b_2^2 - a_1 b_2 \;+ a_1 b_0 b_1$

Apply the following elementary row operations on :

$R_4 \to \; R_4 - K_{12} R_2, \qquad R_5 \to \; R_5 - K_{22} R_2, \qquad R_6 \to \; R_6 - K_{32} R_2, \qquad R_4 \to \; R_4 - K_{13} R_3,$

$R_5 \to \; R_5 - K_{23} R_3, \qquad R_6 \to \; R_6 - K_{33} R_3$

and remove columns 5 and 6, rows 2 and 3, we get a $4 \times$ 4$ matrix β' where

$\beta_{11}' \;= \; a_0, \quad \beta_{12}' \;= \; a_1, \quad \beta_{13}' \;= \; a_2^2, \quad \beta_{14}' \;= \; a_3^3$

$\beta_{21}' \;= \; b_1^2(a_0 b_0^2 \;+ a_1 b_2 \;+ a_1 b_0 b_1 \;+ 2a_2 b_0 b_2) \;+ (b_2 \;- b_0 b_1)(a_2^2 b_0^2 \;+ a_1^2 \;+ a_0 a_3 b_0^2 \;+$

$\quad 2a_1 a_2 b_0 \;+ a_1 a_3 b_2 \;+ a_1 a_3 b_0 b_1 \;+ 2a_2 a_3 b_0 b_2) + \; b_3(a_3 b_3 \;- 2)$

$\beta'_{22} = b_0^2(a_2^2b_0^2 + a_1^2 + a_0a_3b_0^2 + 2a_1a_2b_0 + a_1a_3b_2 + a_1a_3b_0b_1 + 2a_2a_3b_0b_2)$

$-(b_2 + b_0b_1)(a_0b_0^2 + a_1b_2 + a_1b_0b_1 + 2a_2b_0b_2)$

$\beta'_{23} = -(a_0 - 2a_2b_1)(a_0b_0^2 + a_1b_2 + a_1b_0b_1 + 2a_2b_0b_2) - (a_1 + 2a_2b_0)(a_2^2b_0^2 + a_1^2$

$+a_0a_3b_0^2 + 2a_1a_2b_0 + a_1a_3b_2 + a_1a_3b_0b_1 + 2a_2a_3b_0b_2)$

$\beta'_{24} = 2a_1^2a_3 + a_2^2b_2 - a_0a_1 + 3a_1a_3^2b_2 + a_2^2b_0b_1 + 3a_0a_3^2b_0^2 + 2a_2^2a_3b_0^2$

$+2a_1a_2b_1 + 3a_1a_3^2b_0b_1 + 6a_2a_3^2b_0b_2 + 4a_1a_2a_3b_0$

$\beta'_{31} = b_1^2(a_1b_1^2 - a_0b_2 + a_0b_0b_1 + 2a_2b_1b_2) - (b_2 - b_0b_1)(a_2^2b_2 + a_0a_1$

$-a_1a_3b_1^2 - a_2^2b_0b_1 + 2a_0a_2b_0 + a_0a_3b_2 - a_0a_3b_0b_1 - 2a_2a_3b_1b_2)$

$\beta'_{32} = -(b_2 + b_0b_1)(a_1b_1^2 - a_0b_2 + a_0b_0b_1 + 2a_2b_1b_2) - b_0^2(a_2^2b_2 + a_0a_1 - a_1a_3b_1^2$

$-a_2^2b_0b_1 + 2a_0a_2b_0 + a_0a_3b_2 - a_0a_3b_0b_1 - 2a_2a_3b_1b_2) + b_3(a_3b_3 - 2)$

$\beta'_{33} = (a_1 + 2a_2b_0)(a_2^2b_2 + a_0a_1 - a_1a_3b_1^2 - a_2^2b_0b_1 + 2a_0a_2b_0 + a_0a_3b_2$

$-a_0a_3b_0b_1 - 2a_2a_3b_1b_2) - (a_0 - 2a_2b_1)(a_1b_1^2 - a_0b_2 + a_0b_0b_1 + 2a_2b_1b_2)$

$\beta'_{34} = a_2^2b_1^2 + a_0^2 - 3a_0a_3^2b_2 - 2a_2^2a_3b_2 + 3a_1a_3^2b_1^2 - 2a_0a_1a_3 - 2a_0a_2b_1$

$+3a_0a_3^2b_0b_1 + 2a_2^2a_3b_0b_1 + 6a_2a_3^2b_1b_2 - 4a_0a_2a_3b_0$

$\beta'_{41} = b_1^2b_2^2 - (b_2 - b_0b_1)(a_0b_0^2 - a_3b_2^2 - a_1b_2 + a_1b_0b_1)$

$\beta'_{42} = -b_2^2(b_2 + b_0b_1) - b_0^2(a_0b_0^2 - a_3b_2^2 - a_1b_2 + a_1b_0b_1)$

$\beta'_{43} = -b_2^2(a_0 - 2a_2b_1) + b_3(a_3b_3 - 2) + (a_1 + 2a_2b_0)(a_0b_0^2 - a_3b_2^2 - a_1b_2 + a_1b_0b_1)$

$\beta'_{44} = -a_1b_1^2 + 3a_3^2b_2^2 - a_0b_2 - 2a_0a_3b_0^2 + 2a_1a_3b_2 - a_0b_0b_1 - 2a_1a_3b_0b_1$

$det(\beta') = ((a_0b_0 + a_1b_1 + a_2b_2)^2 + a_3b'_3)^3 = 1.$

Thus β' is the completion of (a_0, a_1, a_2^2, a_3^3).

REFERENCES

1. Lam TY. Serre's Conjecture, Lecture Notes in Mathematics, 635, Springer Verlag, New York (1978)
2. Suslin AA. On Stably Free Modules, *Math. USSR Sbornik*, 1977; 31:479 – 491.
3. Suslin AA. On the structure of the Special Linear Group over Polynomial rings, *Math. USSR Izvestija*, 1977; 11 :221 - 238.
4. Suslin AA and Vaserstein LN. Serre's problem on Projective Modules over Polynomial Rings and Algebraic K-theory, *Math. USSRIzvestija*, 1976; 10: 937 - 1001.
5. Swan RG and Towber J. A class of projective modules which are nearly free, *Journal of Algebra*, 1975; 36: 427–434.