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RESEARCH ARTICLE

Algorithmic Approach to the center of Special Unimodular Vector Group

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ABSTRACT

In this paper, we write an algorithm to construct Suslin matrices and show that center of the Special Unimodular Vector Group of scalar matrices uI_{2^n}.

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1. INTRODUCTION

Let R be a commutative ring with 1. Swan and Towber (1975) proved that *If P is a projective R[X]-module of rank 2 and* $X^2R[X]^2 \subseteq P \subseteq R[X]^2$, *then* $P \simeq R[X]^2$. As a consequence, they concluded that if $(a,b,c) \in Um_3(R)$, then (a^2,b,c) can be completed to an invertible matrix. This result was explained and generalized by Suslin in his doctoral thesis (Suslin, 1977) in the mid-seventies. There he proves that *if* $(a_0,a_1,\dots,a_r) \in Um_{r+1}(R)$, *then the unimodular row* $(a_0,a_1,a_2^2,\dots,a_r^r)$ *can always be completed to an invertible matrix.*

Suslin (1977) also describes an inductive (on *r*) method of constructing the completion. Given a pair of row s$v,w \in M_{1\,r+1}(R), r \geq 0$ he defines a matrix $S_r(v,w) \in M_{2^r}(R)$ with determinant equal to the inner product $\langle v,w \rangle = v \cdot w^T$.

2. Definition of Suslin Matrices

The construction of the Suslin matrix $S_r(v,w)$ is possible once we have two rows *v, w* of length *r*+1, $r \geq 0$. A.A. Suslin gave an inductive process to construct $S_r(v,w)$ in Suslin (1977). We elaborate on it here.

Definition 2.1 Let $v = (a_0, a_1, \dots, a_r) = (a_0, v_1)$, $w = (b_0, b_1, \dots, b_r) = (b_0, w_1) \in M_{1\,r+1}(R)$, where $v_1 = (a_1, \dots, a_r)$ and $w_1 = (b_1, \dots, b_r)$. Set $S_0(v,w) = a_0$, and

$$S_r(v,w) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ S_{r-1}(w_1, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}$$

where the superscript T stands for the transpose of a matrix.

We call $S_r(v,w)$ the *Suslin matrix associated to the pair (v,w)*. $S_r(v,w)$is said to be *special* if $\langle v,w \rangle = 1$.

Thus beginning with $S_0((a_0),(b_0)) = a_0$, we have

$$S_1((a_0,a_1),(b_0,b_1)) = \begin{pmatrix} a_0 & a_1 \\ -b_1 & b_0 \end{pmatrix}$$

$$S_2((a_0,a_1,a_2),(b_0,b_1,b_2)) = \begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix}$$

and so on.

Note that $S_r(v,w)$ is a matrix of order 2^r and has only 0, $\pm a_i$, and $\pm b_i$ $(0 \le i \le r)$ as its entries, and there are at most *r*+1 non-zero elements in each row and in each column.

By observation, one gets the positions of a_i and b_i in $S_r(v,w)$ as follows: For $1 \le i \le r-1$,

1. The positions of a_0 in $S_r(v,w)$ is given by (k,k), $1 \le k \le 2^{r-1}$ and the positions of b_0 in $S_r(v,w)$ is given by (k,k), $2^{r-1}+1 \le k \le 2^r$.
2. The positions of a_r in $S_r(v,w)$ is given by$(2k-1, 2^r-2k+2)$, $1 \le k \le 2^{r-1}$and the positions of b_r in $S_r(v,w)$ is given by $(2k, 2^r-2k+1)$, $1 \le k \le 2^{r-1}$.
3. The positions of a_i in $S_r(v,w)$ is given by $(2^2k\, 2^{r-1-i} + j, (2 + (2^{i-1}-k-1)2^2)2^{r-1-i} + j)$, where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$.
4. The positions of $-a_i$ in $S_r(v,w)$ is given by $((3+2^2k)2^{r-1-i} + j, (1 + (2^{i-1}-k-1)2^2)2^{r-1-i} + j)$, where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$.
5. The positions of b_i in $S_r(v,w)$ is given by $((1 + (2^{i-1}-k-1)2^2)2^{r-1-i} + j, (3+2^2k)2^{r-1-i} + j)$, where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$.
6. The positions of $-b_i$ in $S_r(v,w)$ is given by $(2 + (2^{i-1}-k-1)2^2)2^{r-1-i} + j, 2^2k\, 2^{r-1-i} + j)$, where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$.

3. Elementary Matrices

In the $r \times r$\$ matrices there are r^2 particular matrices that play a key role. These are called the *matrix units*, e_{ij},which are defined as follows: e_{ij} is the matrix whose *ij*-thentry is 1 and all other entries are 0.

Definition 3.1 *The **General Linear** group $GL_r(R)$ is defined as the group of $r \times r$ invertible matrices with entries in R.*

Definition 3.2 *The **Special Linear** group is denoted by $SL_r(R)$ and is defined as*
$SL_r(R) = \{\alpha \in GL_r(R) : det(\alpha) = 1\}$.

Definition 3.3 *The group of **elementary matrices** $E_r(R)$ isa subgroup of $GL_r(R)$ generated by matrices of the form $E_{ij}(\lambda) = I_r + \lambda e_{ij}$, where $\lambda \in R, i \neq j$ and $e_{ij} \in M_r(R)$ with ij-th entry is* 1 *and all other entries are* 0*.*

Following are some well-known properties of the elementary generators:

Lemma 3.4 For $\lambda, \mu \in R$,
1. (Splitting Property) $E_{ij}(\lambda + \mu) = E_{ij}(\lambda)E_{ij}(\mu), 1 \leq i \neq j \leq r$.
2. (Commutator Law) $[E_{ij}(\lambda), E_{jk}(\mu)] = E_{ik}(\lambda\mu), 1 \leq i \neq j \neq k \leq r$.

Remark 3.5 In view of the Commutator Law, $E_r(R)$ is generated by $\{E_{1i}(\lambda), E_{i1}(\mu) : 2 \leq i \leq r, \lambda, \mu \in R\}$.

As *R* is commutative, $E_{ij}(\lambda), i \neq j, \lambda \in R$, is invertible with inverse $E_{ij}(-\lambda)$. In fact, $E_{ij}(\lambda)$ belongs to $SL_r(R)$. Hence, $E_r(R) \subseteq SL_r(R) \subseteq GL_r(R)$.

Definition 3.6 *A row $v = (v_1, v_2, \ldots, v_r) \in R^r$ is said to be **unimodular** (of length r) if there exists elements $w_1, w_2, \ldots, w_r$ in R such that $v_1w_1 + v_2w_2 + \cdots + v_rw_r = 1$. $Um_r(R)$ will denote the set of all unimodular rows $v \in R^r$.*

4. The Unimodular Vector Groups

We decide to play with Suslin matrices $S_r(v,w)$ and study a subgroup of $SL_{2^r}(R)$, which is defined as:

Definition 4.1 *The **Special Unimodular Vector group** $SUm_r(R)$ is the subgroup of $SL_{2^r}(R)$ generated by the Suslin matrices $S_r(v,w)$ w.r.t. the pair (v, w), with $v \in Um_{r+1}(R)$, and for some w with $\langle v, w \rangle = 1$.*

Analogous to the Elementary subgroup $E_r(R)$ of $GL_r(R)$ we consider the Elementary Unimodular Vector subgroup, which is defined as:

Definition 4.2 *The Elementary Unimodular Vector group $EUm_r(R)$ is the subgroup of $SUm_r(R)$ generated by the Suslin matrices $S_r(v,w)$, with $v \in e_1E_{r+1}(R)$, and with $\langle v, w \rangle = 1$.*

Definition 4.3 *We shall denote by $EUm_r(R)^*$ the subgroup of $EUm_r(R)$ generated by the elements* $S\_r(e_1E_{1i}(\lambda), e_1), S_r(e_1, e_1E_{1i}(\lambda))$, *with* $2 \leq i \leq r+1, \lambda \in R$.

Notation 4.4 For a matrix $\alpha \in M_k(R)$, we define α^{top} as the matrix whose entries are the same as that of α above the diagonal, and on the diagonal, and is zero below the diagonal. Similarly, we define α^{bot}.

For simplicity we may write α^t for α^{top}, α^b for α^{bot} and α^T for α transpose. Moreover, we use α^{tb} for α^{top} or α^{bot}.

Definition 4.5 *We shall denote by $EUm_r(R)^{tb}$ the subgroup of $E_{2^r}(R)$ generated by the elements α^{tb} for α a basic generator of $EUm_r(R)^*$.*

In (2005) Jose – Rao proved the following:

Proposition 4.6 *$EUm_r(R)^{tb}$ is a subgroup of $EUm_r(R)$; and, in fact, equals it.*

5. Center of $SU\, m_r\,(R)$

In this section we calculate algorithmically the center of the Special Unimodular vector group $SUm_r(R)$. We begin with some lemmae.

Lemma 5.1 *Let $A \in M_{2^s}\left(M_{2^t}(R)\right)$, $t \geq 1$, $s + t = r$ be a diagonal block matrix, where the alternating diagonal blocks are the same. If A commutes with $E_r(e_{s+1})(1)^{top}$ and $E_r(e^*_{s+1})(1)^{top}$ then $A \in M_{2^{s+1}}(M_{2^{t-1}}(R))$ is a diagonal block matrixwhose alternating diagonal block entries are same.*

Proof: Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \in M_{2^{t-1}}(M_2(R))$ be the two,perhaps different, diagonal blocks of A. Compare the $(1, 2^s)$-th,$(1, 2^s - 1)$-th, and $(2, 2^s - 1)$-th block entries of $A\,E_r(e_{s+1})(1)^{top}$and $E_r(e_{s+1})(1)^{top}A$ we get $a_{21} = 0,\ a_{34} = 0$, and $a_{11} = a_{33}$ respectively. Compare the$(1, 2^s)$-th, $(2, 2^s)$-th,and $(2, 2^s - 1)$-th block entries of$A\,E_r(e^*_{s+1})(1)^{top}$and $E_r(e^*_{s+1})(1)^{top}A$ we get$a_{12} = 0, a_{22} = a_{44}$, and a_43 $= 0$ respectively. Hence$A \in M_{2^{s+1}}(M_{2^{t-1}}(R))$ and is a diagonalmatrix with alternating entries equal.

Lemma 5.2 *Let $A \in M_{2^r}(R)$ be a diagonal matrix with equal alternating diagonal entries. If A commutes with $E_r(e_{r+1})(1)^{top}$ then A is a scalar matrix.*

Proof: Let a_{11} and a_{22} be the two differentdiagonal entries of the matrix A. Compare the $(1, 2^r)$-th entryof $AE_r(e_{r+1})(1)^{top}$and $E_r(e_{r+1})(1)^{top}A$, we get$a_{11} = a_{22}$. Hence A is a scalar matrix.

Proposition 5.3 (Center of $SUm_r(R)$) *Let$A \in M_{2^r}(R)$. If A commutes with every element of $SUm_r(R)$, then A is a scalar matrix.*

Proof: Since $SUm_2(R) = SL_2(R)$, the result isclear for $r = 1$. So let $r \geq 2$. Let us write $A = \left(a_{ij}\right)_{1\leq i,j\leq 4}$ in block form. By comparing entries we observe that

1. $E_r(e^*_2)(1)^{top}A = A\,E_r(e^*_2)(1)^{top}$ implies $a_{12} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} = a_{44}$,
2. $E_r(e_2)(1)^{top}A = A\,E_r(e_2)(1)^{top}$ implies $a_{21} = a_{31} = a_{34} = 0, a_{11} = a_{33}$,
3. $E_r(e^*_2)(1)^{bot}A = A\,E_r(e^*_2)(1)^{bot}$ implies $a_{13} = a_{14} = a_{23} = 0$, and
4. $E_r(e_2)(1)^{bot}A = A\,E_r(e_2)(1)^{bot}$implies $a_{24} = 0$.

Hence $A \in M_{2^2}(M_{2^{r-2}}(R))$ is adiagonal block matrix with alternating diagonal blocks same. ApplyLemma 5.1$r - 2$ times and conclude that A is a diagonalmatrix with alternating entries same. Now apply Lemma 5.2 toget the desired result.

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