RESEARCH ARTICLE

Linear transformation of the action of $SL_n(R)$ on the space of alternating matrices

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ABSTRACT

In this paper, we compute the linear transformation associated tothe action of the special linear groups on the space of all alternating matrices.

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1. INTRODUCTION

A famous theorem of Vaserstein in [(Vaserstein, 1987) Theorem 5.2 and Corollary 7.4] states that the orbit space $Um_3(R)/E_3(R)$ of uni modular rows under elementary action is in bijective correspondence to the elementary symplectic Witt group $W_E(R)$, when R is a commutative ring of Krull dimension two.(Recall that $W_E(R)$ is the group of stably equivalent alternating matrices of P fafian one over R.)

To prove this theorem Vaserstein (1987) evolves the study of the elementary group on an invertible alternating matrix.

2. PRELIMINARIES

Let R be a commutative ring with 1. A matrix $A \in M_n(R)$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$, for $1 \leq i,j \leq n$. The space of all alternating $n \times n$ matrices over a commutative ring R will be denoted by $Alt_n(R)$. It is clearly a free R-module of rank $1 + 2 + \cdots + (n-1) = \binom{n}{2}$ with basis $B_{ij} = e_{ij} - e_{ji}$, $1 \leq i < j \leq n$, where $e_{ij} \in M_n(R)$ with *ij*-th entry is 1 and all other entries are 0.

Definition 2.1 *The **General Linear** group $GL_r(R)$ is defined as the group of $r \times r$ invertible matrices with entries in R.*

Definition 2.2 *The **Special Linear** group is denoted by $SL_r(R)$ and is defined as*
$SL_r(R) = \{\alpha \in GL_r(R) : det(\alpha) = 1\}$.

Definition 2.3 *The group of **elementary matrices** $E_r(R)$ is a subgroup of $GL_r(R)$ generated by matrices of the form $E_{ij}(\lambda) = I_r + \lambda e_{ij}$, where $\lambda \in R, i \neq j$ and $e_{ij} \in M_r(R)$ with ij-th entry is* 1 *and all other entries are* 0.

Following are some well-known properties of the elementary generators:

Lemma 2.4 For $\lambda, \mu \in R$,

1. (Splitting Property) $E_{ij}(\lambda + \mu) = E_{ij}(\lambda)E_{ij}(\mu), 1 \leq i \neq j \leq r$.
2. (Commutator Law) $[E_{ij}(\lambda), E_{jk}(\mu)] = E_{ik}(\lambda\mu), 1 \leq i \neq j \neq k \leq r$.

Remark 2.5 In view of the Commutator Law, $E_r(R)$ is generated by
$\{E_{1i}(\lambda), E_{i1}(\mu) : 2 \leq i \leq r, \lambda, \mu \in R\}$.

As *R* is commutative, $E_{ij}(\lambda), i \neq j, \lambda \in R$, is invertible with inverse $E_{ij}(-\lambda)$. In fact, $E_{ij}(\lambda)$ belongs to $SL_r(R)$. Hence, $E_r(R) \subseteq SL_r(R) \subseteq GL_r(R)$.

3. COMPOUND MATRICES

In this section we see the definition and properties of Compound matrices. We begin with some basic definitions:

Definition 3.1 (Minors of a matrix) *Given an $n \times m$ matrix $A = (a_{ij})$, a minor of A is the determinant of a smaller matrix formed from its entries by selecting only some of the rows and columns.*
Let $K = \{k_1, k_2, \dots, k_p\}$ and $L = \{l_1, l_2, \dots, l_p\}$ be subsets of $\{1,2,\dots,n\}$ and $\{1,2,\dots,m\}$, respectively. The indices are chosen such that $k_1 < k_2 < \cdots < k_p$ and $l_1 < l_2 < \cdots < l_p$. The p-th order minor defined by K and L is the determinant of the submatrix of A obtained by considering the rows $k_1, k_2, \dots, k_p$ and columns $l_1, l_2, \dots, l_p$ of A. We denote this submatrix as $A\begin{pmatrix} k_1 & k_2\cdots & k_p \\ l_1 & l_2\cdots & l_p \end{pmatrix}$.

We now state a well-known theorem:

Theorem 3.2 (The Cauchy-Binet formula) *Let A be a $m \times n$ matrix and B a $n \times m$ matrix. Then the determinant of their product $C = AB$ can be written as a sum of products of minors of A and B, i.e.*

$$|C| = \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} A\begin{pmatrix} 1 & 2 & \cdots m \\ k_1 & k_2 & \cdots k_m \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \cdots k_m \\ 1 & 2 & \cdots m \end{pmatrix}$$

The sum is over the maximal (m-th order) minors of A and the corresponding minor of B. In particular, det(*AB*) = det(*A*)det(*B*), *if A, B are $n \times n$ matrices.*

First recall the notion of the compound matrix:

Definition 3.3 *Suppose that A is an $m \times n$ matrix with entries from a ring R and $1 \leq r \leq min(m,n)$. The r^{th} compound matrix $C_r(A)$ or r^{th} adjugate of A is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose entries are the minors of order r, arranged in lexicographic order, i.e.*

$$C_r(A) = \left(\begin{pmatrix} i_1 & i_2 & \cdots i_r \\ j_1 & j_2 & \cdots j_r \end{pmatrix}\right)$$

Following are some properties of Compound matrices.

Lemma 3.4 (Properties) [1] *Let A and B be $n \times n$ matrices and $r \leq n$. Then*

1. $C^1(A) = A$
2. $C^n(A) = det(A)$
3. $C^r(AB) = C^r(A)C^r(B)$
4. $C^r(A^t) = \left(C^r(A)\right)^t$

4. ASSOCIATED LINEAR TRANSFORMATIONS

In this section, we find the linear transformation of the action of $SL_n(R)$ on the space $Alt_n(R)$ of alternating matrices.

One can define the action of $SL_n(R)$ on $Alt_n(R)$ as

$SL_n(R) \times Alt_n(R) \to Alt_n(R)$
$(\sigma, A) \mapsto \sigma A \sigma^t$
This action enables one to associate a linear transformation $T_\sigma : Alt_n(R) \to Alt_n(R)$ for $\sigma \in SL_n(R)$, via $T_\sigma(A) = \sigma A \sigma^t$.

Let us compute T_σ, $\sigma \in SL_n(R)$. We prove that it is the matrix of $\wedge^2 \sigma$.

Lemma 4.1 *Let $\sigma: R^n \to R^m$ be a R-linear map. Then the matrix of the linear transformation $\wedge^r \sigma: \wedge^r R^n \to \wedge^r R^m$ is $C_r(M(\sigma))$, where $M(\sigma)$ is the matrix of σ .*

Proof: This is well-known to experts when R is a field. We compute it as follows:

Let $e_1, \ldots, e_n$ be a basis of R^n. and $f_1, \ldots, f_m$ be a basis of R^m. Let us compute the matrix of $\wedge^r \sigma$ w.r.t. the standard basis $e_{i_1} \wedge \ldots \wedge e_{i_r}$ ordered lexico graphically, and $f_{i_1} \wedge \ldots \wedge f_{i_r}$ ordered lexico graphically. Suppose $1 \leq i_1 < \cdots < i_r \leq n$ as usual. Then

$$\wedge^r(\sigma)\left(e_{i_1} \wedge \ldots \wedge e_{i_r}\right) = \sigma\left(e_{i_1}\right) \wedge \ldots \wedge \sigma\left(e_{i_r}\right)$$

$$\sum_{j=1}^{m} d_{ji_1} f_j \wedge \ldots \wedge \sum_{j=1}^{m} d_{ji_r} f_j = \sum_{1 \leq j_1 < \cdots < j_r \leq n} A\begin{pmatrix} j_1 & j_2 & \cdots j_r \\ i_1 & i_2 & \cdots i_r \end{pmatrix} f_{j_1} \wedge \ldots \wedge f_{j_r}$$

Where A denotes the matrix of the linear transformation σ.

Theorem 4.2 The matrix of the linear transformation T_σ is the same as the matrix of the linear transformation $\wedge^2 \sigma : \wedge^2 R^n \to \wedge^2 R^n$; which is the compound matrix of order 2 associated to σ.

Proof: Follows from Lemma 4.1.

The following Theorem gives the explicit formula for $[T_\sigma]$, where $\sigma = E_{1i}(\lambda), E_{i1}(\lambda)$.

Theorem 4.3 *Let $\sigma = E_{1i}(\lambda)$ or $E_{i1}(\lambda), 2 \le i \le n, \lambda \in R$, the basic generators of $E_n(R)$. Then the matrix of T_σ with respect to the ordered basis $\{B_{12}, B_{13}, \dots, B_{1n}, B_{23}, B_{24}, \dots, B_{2n}, B_{34}, \dots, B_{n-1,n}\}$ of $Alt_n(R)$ is*

1. *If $i = 2$, then $[T_{E_{12}}(\lambda)] = I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=i}^{n-1} e_{j,n-2+j}$*
2. *If $i = 3$, then $[T_{E_{13}}(\lambda)] = I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=i}^{n-1} e_{j,2n-5+j} - \lambda e_{1,n}$*
3. *For $1 \le i \le n-1$,*
$$[T_{E_{1i}}(\lambda)] = I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=i}^{n-1} e_{j,n-i+j+\sum_{k=2}^{i-1}(n-k)} - \lambda e_{1,n+i-3} - \lambda \sum_{j=2}^{i-2} e_{j,n+i-3+\sum_{k=3}^{j+1}(n-k)}$$
4. *If $i = n$, then $[T_{E_{12}}(\lambda)] = I_{\frac{n(n-1)}{2}} - \lambda e_{1,n+i-3} - \lambda \sum_{j=2}^{i-2} e_{j,n+i-3+\sum_{k=3}^{j+1}(n-k)}$*
   and $[T_{E_{i1}}(\lambda)] = [T_{E_{1i}}(\lambda)]^T$

Proof: When $\sigma = E_{12}(\lambda)$, by Theorem 4.2, $[T_\sigma] = \left(\sigma\begin{pmatrix} i & j \\ k & l\end{pmatrix}\right)_{1\le i<j,k<l\le n}$

Note that for $3 \le r \le n$, $\sigma\begin{pmatrix} 1 & r \\ 2 & r\end{pmatrix} = \lambda$, for $1 \le s < r \le n$, $\sigma\begin{pmatrix} s & r \\ s & r\end{pmatrix} = 1$ and all other entries are zero. Thus we have,

$$\begin{aligned}[T_{E_{12}}(\lambda) &= \begin{pmatrix} I_{n-1} & \begin{matrix} 0 & 0 \\ 0 & \lambda I_{n-2}\end{matrix} & 0_{n-1\times\frac{(n-2)(n-3)}{2}} \\ 0_{\frac{(n-1)(n-2)}{2}\times(n-1)} & & I_{\frac{(n-1)(n-2)}{2}}\end{pmatrix} \\ &= I_{\frac{n(n-1)}{2}} + \lambda(e_{2,n} + e_{3,n+1} + \cdots + e_{n-1,2n-3}) \\ &= I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=2}^{n-1} e_{j,n-2+j}\end{aligned}$$

When $\sigma = E_{13}(\lambda)$, note that $\sigma\begin{pmatrix} 1 & 2 \\ 2 & 3\end{pmatrix} = -\lambda$, for $4 \le r \le n$, $\sigma\begin{pmatrix} 1 & r \\ 3 & r\end{pmatrix} = \lambda$, for $1 \le s < r \le n$, $\sigma\begin{pmatrix} s & r \\ s & r\end{pmatrix} = 1$ and all other entries are zero. Thus we have,

$$[T_{E_{13}}(\lambda)] = I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=i}^{n-1} e_{j,2n-5+j} - \lambda e_{1,n}$$

When $\sigma = E_{1i}(\lambda)$, $4 \le i \le n-1$, note that for $2 \le r \le i-1$, $\sigma\begin{pmatrix} 1 & r \\ r & i\end{pmatrix} = -\lambda$, for $i = 1 \le r \le n$, $\sigma\begin{pmatrix} 1 & r \\ i & r\end{pmatrix} = \lambda$, for $1 \le s < r \le n$, $\sigma\begin{pmatrix} s & r \\ s & r\end{pmatrix} = 1$ and all other entries are zero. Thus we have,

$$[T_{E_{1i}}(\lambda)] = I_{\frac{n(n-1)}{2}} + \lambda \sum_{j=i}^{n-1} e_{j,n-i+j+\sum_{k=2}^{i-1}(n-k)} - \lambda e_{1,n+i-3} - \lambda \sum_{j=2}^{i-2} e_{j,n+i-3+\sum_{k=3}^{j+1}(n-k)}$$

The case when $\sigma = E_{1n}(\lambda)$ can be proved similarly. $[T_{E_{i1}}(\lambda)] = [T_{E_{1i}}(\lambda)]^T$ follows from Lemma 3.4 (iv).

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