

# On Semi-symmetric Para Kenmotsu Manifolds

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**Abstract** In this paper we study some remarkable properties of para Kenmotsu (briefly  $p$ -Kenmotsu) manifolds satisfying the conditions  $R(X, Y).R = 0$ ,  $R(X, Y).P = 0$  and  $P(X, Y).R = 0$ , where  $R(X, Y)$  is the Riemannian curvature tensor and  $P(X, Y)$  is the Weyl projective curvature tensor of the manifold. It is shown that a semi-symmetric  $p$ -Kenmotsu manifold  $(M_n, g)$  is of constant curvature and hence is an  $sp$ -Kenmotsu manifold. Also, we obtain the necessary and sufficient condition for a  $p$ -Kenmotsu manifold to be Weyl projective semi-symmetric and shown that the Weyl projective semi-symmetric  $p$ -Kenmotsu manifold is projectively flat. Finally we prove that if the condition  $P(X, Y).R = 0$  is satisfied on a  $p$ -Kenmotsu manifold then its scalar curvature is constant.

**Keywords:** para Kenmotsu manifolds, curvature tensor, projective curvature tensor, scalar curvature

**Cite This Article:** T. Satyanarayana, and K. L. Sai Prasad, "On Semi-symmetric Para Kenmotsu Manifolds." *Turkish Journal of Analysis and Number Theory*, vol. 3, no. 6 (2015): 145-148. doi: 10.12691/tjant-3-6-1.

## 1. Introduction

The notion of an almost para-contact Riemannian manifold was introduced by Sato [7] in 1976. After that, T. Adati and K. Matsumoto [1] defined and studied  $p$ -Sasakian and  $sp$ -Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [6] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [9] have defined a class of almost para-contact metric manifolds namely para-Kenmotsu (briefly  $p$ -Kenmotsu) and special para Kenmotsu (briefly  $sp$ -Kenmotsu) manifolds. In a recent paper, the authors Satyanarayana and Sai Prasad [8] studied conformally symmetric  $p$ -Kenmotsu manifolds, that is the  $p$ -Kenmotsu manifolds satisfying the condition  $R(X, Y).C = 0$ , and they prove that such a manifold is conformally flat and hence is an  $sp$ -Kenmotsu manifold, where  $R$  is the Riemannian curvature and  $C$  is the conformal curvature tensor defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \left[ g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \right] + \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y]. \quad (1.1)$$

Here  $S$  is the Ricci tensor,  $r$  is the scalar curvature and  $Q$  is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor  $S$  [3] i.e.,

$$g(QX, Y) = S(X, Y). \quad (1.2)$$

A Riemannian manifold  $M$  is locally symmetric if its curvature tensor  $R$  satisfies  $\nabla R = 0$ , where  $\nabla$  is Levi-Civita connection of the Riemannian metric [4]. As a

generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold  $M_n$  is said to be semi-symmetric if its curvature tensor  $R$  satisfies  $R(X, Y).R = 0$  where  $R(X, Y)$  acts on  $R$  as derivation [10]. Locally symmetric and semi-symmetric  $p$ -Sasakian manifolds are widely studied by many geometers [2,5].

In this study, we consider the  $p$ -Kenmotsu manifolds satisfying the conditions  $R(X, Y).R = 0$ , known as semi-symmetric  $p$ -Kenmotsu manifolds, where  $R(X, Y)$  is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors  $X$  and  $Y$  and the  $p$ -Kenmotsu manifolds  $(M_n, g)$  ( $n > 2$ ) satisfying the condition  $R(X, Y).P = 0$ , where  $P$  denotes the Weyl projective curvature tensor [12] defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \left[ g(Y, Z)QX - g(X, Z)QY \right]. \quad (1.3)$$

Here we consider the  $p$ -Kenmotsu manifolds  $M_n$  for  $n > 2$ ; as if for  $n = 2$ , the projective curvature tensor identically vanishes.

In section 3, it is shown that a semi-symmetric  $p$ -Kenmotsu manifold  $(M_n, g)$  of constant curvature is an  $sp$ -Kenmotsu manifold. In the next section we obtain the necessary and sufficient condition for a  $p$ -Kenmotsu manifold to be Weyl projective semi-symmetric and shown that the Weyl projective semi-symmetric  $p$ -Kenmotsu manifold is projectively flat. Finally we prove that if the condition  $P(X, Y).R = 0$  is satisfied on a  $p$ -Kenmotsu manifold then its scalar curvature is constant.

## 2. $p$ -Kenmotsu Manifolds

Let  $M_n$  be an  $n$ -dimensional differentiable manifold equipped with structure tensors  $(\Phi, \xi, \eta)$  where  $\Phi$  is a

tensor of type (1,1),  $\xi$  is a vector field,  $\eta$  is a 1-form such that

$$\eta(\xi) = 1 \tag{2.1}$$

$$\Phi^2(X) = X - \eta(X)\xi; \overline{X} = \Phi X. \tag{2.2}$$

Then  $M_n$  is called an almost para contact manifold.

Let  $g$  be the Riemannian metric in an  $n$ -dimensional almost para-contact manifold  $M_n$  such that

$$g(X, \xi) = \eta(X) \tag{2.3}$$

$$\Phi \xi = 0, \eta(\Phi X) = 0, \text{rank } \Phi = n - 1 \tag{2.4}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all vector fields  $X$  and  $Y$  on  $M_n$ . Then the manifold  $M_n$  [7] is said to admit an almost para-contact Riemannian structure  $(\Phi, \xi, \eta, g)$  and the manifold is called an almost para-contact Riemannian manifold.

A manifold of dimension ' $n$ ' with Riemannian metric ' $g$ ' admitting a tensor field ' $\Phi$ ' of type (1, 1), a vector field ' $\xi$ ' and a 1-form ' $\eta$ ' satisfying (2.1), (2.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{2.6}$$

$$\begin{aligned} (\nabla_X \nabla_Y \eta)Z &= [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) \\ &+ [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z) \end{aligned} \tag{2.7}$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \tag{2.8}$$

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X \tag{2.9}$$

is called a para-Kenmotsu manifold or briefly  $p$  - Kenmotsu manifold [9].

A  $p$  -Kenmotsu manifold admitting a 1-form ' $\eta$ ' satisfying

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{2.10}$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \phi(\overline{X}, Y), \tag{2.11}$$

where  $\phi$  is an associate of  $\Phi$ ,

is called a special  $p$  -Kenmotsu manifold or briefly  $sp$  - Kenmotsu manifold [9].

It is known that [9] in a  $p$  -Kenmotsu manifold the following relations hold:

$$S(X, \xi) = -(n-1)\eta(X) \text{ where } g(QX, Y) = S(X, Y) \tag{2.12}$$

$$\begin{aligned} g[R(X, Y)Z, \xi] &= \eta[R(X, Y, Z)] \\ &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \end{aligned} \tag{2.13}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \tag{2.14}$$

$$\begin{aligned} R(X, Y, \xi) &= \eta(X)Y - \eta(Y)X; \\ \text{when } X &\text{ is orthogonal to } \xi \end{aligned} \tag{2.15}$$

where  $S$  is the Ricci tensor and  $R$  is the Riemannian curvature.

Moreover, it is also known that if a  $p$  -Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value  $-n(n-1)$ . Especially, if a  $p$  -Kenmotsu manifold is of

constant curvature, the scalar curvature has a negative constant value  $-n(n-1)$  [9]. In this case,

$$S(Y, Z) = -(n-1)g(Y, Z) \tag{2.16}$$

and hence

$$S(\Phi Y, \Phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z). \tag{2.17}$$

Also, if a  $p$  -Kenmotsu manifold is of constant curvature, we have

$$'R(X, Y, Z, P) = \frac{1}{(n-1)} \begin{bmatrix} S(Y, Z)g(X, P) \\ -S(X, Z)g(Y, P) \end{bmatrix}. \tag{2.18}$$

The above results will be used further in the next sections.

### 3. $p$ -Kenmotsu Manifolds Satisfying $R(X, Y).R = 0$

In this section, we consider semi-symmetric  $p$  - Kenmotsu manifolds, i.e.,  $p$  -Kenmotsu manifolds satisfying the conditions  $R(X, Y).R = 0$  where  $R(X, Y)$  is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors  $X$  and  $Y$ . Now

$$\begin{aligned} (R(X, Y) \cdot R)(U, V)W &= R(X, Y)R(U, V)W - R(R(X, Y)U, V)W \\ &- R(U, R(X, Y)V)W - R(U, V)R(X, Y)W. \end{aligned} \tag{3.1}$$

Putting  $X = \xi$  in (3.1), and on using the condition  $R(X, Y).R = 0$ , we get

$$\begin{aligned} g(R(\xi, Y)R(U, V)W, \xi) &- g(R(R(\xi, Y)U, V)W, \xi) \\ -g(R(U, R(\xi, Y)V)W, \xi) &- g(R(U, V)R(\xi, Y)W, \xi) \\ &= 0. \end{aligned} \tag{3.2}$$

By using the equations (2.3) and (2.14), from (3.2) we get

$$\begin{aligned} 'R(U, V, W, Y) - \eta(Y)\eta(R(U, V)W) &+ \eta(U)\eta(R(Y, V)W) + \eta(V)\eta(R(U, Y)W) \\ + \eta(W)\eta(R(U, V)Y) - g(Y, U)\eta(R(\xi, V)W) &- g(Y, V)\eta(R(U, \xi)W) - g(Y, W)\eta(R(U, V)\xi) = 0 \end{aligned} \tag{3.3}$$

where  $'R(U, V, W, Y) = g(R(U, V)W, Y)$ .

On putting  $Y = U$  in (3.3), we get

$$\begin{aligned} 'R(U, V, W, U) + \eta(V)\eta(R(U, U)W) &+ \eta(W)\eta(R(U, V)U) - g(U, U)\eta(R(\xi, V)W) \\ -g(U, V)\eta(R(U, \xi)W) - g(U, W)\eta(R(U, V)\xi) &= 0. \end{aligned} \tag{3.4}$$

Now putting  $U = e_i$ , where  $\{e_i\}, i = 1, 2, \dots, n$  is an orthogonal basis of the tangent space at any point, and taking the summation of (3.4) over  $i, 1 \leq i \leq n$ , we get (2.16).

Also, using the equations (2.12), (2.16) and (3.3) we get (2.18), shows that the manifold is of constant curvature.

Thus we state the following result.

**Theorem 3.1:** A semi-symmetric  $p$  -Kenmotsu manifold is of constant curvature.

Now, from (2.16) and (2.18) we have

$$R(X, Y, Z, P) = g(X, Z)g(Y, P) - g(Y, Z)g(X, P), \quad (3.5)$$

and from equations (2.16) and (2.5), we have

$$S(\Phi X, \Phi Y) = -(n-1)[g(X, Y) - \eta(X)\eta(Y)]. \quad (3.6)$$

On contraction of (3.6) with covariant tensor  $\phi(X, Y) = g(\bar{X}, Y)$ , we get

$$\phi(\bar{X}, Y) = g(X, Y) - \eta(X)\eta(Y),$$

shows that the manifold is an  $sp$ -Kenmotsu one.

Thus, we state the following theorem.

**Theorem 3.2:** If a semi-symmetric  $p$ -Kenmotsu manifold  $(M_n, g)$  is of constant curvature, the manifold is an  $sp$ -Kenmotsu one.

#### 4. $p$ -Kenmotsu Manifolds Satisfying $R(X, Y).P = 0$

In this section, we consider Weyl projective semi-symmetric  $p$ -Kenmotsu manifolds, i.e.,  $p$ -Kenmotsu manifolds satisfying the condition  $R(X, Y).P = 0$ . Now

$$\begin{aligned} &(R(X, Y) \cdot P)(U, V)W \\ &= R(X, Y)P(U, V)W - P(R(X, Y)U, V)W \\ &\quad - P(U, R(X, Y)V)W - P(U, V)R(X, Y)W. \end{aligned} \quad (4.1)$$

Put  $X = \xi$  in (4.1). Then the condition  $R(X, Y).P = 0$  implies that

$$\begin{aligned} &g(R(\xi, Y)P(U, V)W, \xi) - g(P(R(\xi, Y)U, V)W, \xi) \\ &- g(P(U, R(\xi, Y)V)W, \xi) - g(P(U, V)R(\xi, Y)W, \xi) \\ &= 0. \end{aligned} \quad (4.2)$$

Then on using equations (2.12), (2.13) and (1.3), we get

$$\eta(P(X, Y)Z) = 0. \quad (4.3)$$

On the other hand, by using (2.3), (2.4), and (4.3), we get

$$g(R(\xi, Y)P(U, V)W, \xi) = -g(P(U, V)W, Y). \quad (4.4)$$

Then from equations (4.2) and (4.3), the left hand side of (4.4) is zero, gives that  $g(P(U, V)W, Y) = 0$  for all  $U, V, W$  and  $Y$  and hence  $P(X, Y) = 0$ . This leads to the following theorem:

**Theorem 4.1:** A Weyl projective semi-symmetric  $p$ -Kenmotsu manifold is projectively flat.

But it is known that [11], a projectively flat Riemannian manifold is of constant curvature. Also it can be easily seen that a manifold of constant curvature is projectively flat. Hence we have the following theorem.

**Theorem 4.2:** A  $p$ -Kenmotsu manifold is Weyl projective semi-symmetric if and only if the manifold is of constant curvature.

Also it is known that a  $p$ -Kenmotsu manifold of constant curvature is an  $sp$ -Kenmotsu manifold [8]. Hence we conclude the following result:

**Theorem 4.3:** A Weyl projective semi-symmetric  $p$ -Kenmotsu manifold is of constant curvature and hence is an  $sp$ -Kenmotsu manifold.

It is trivial that in case of a projective symmetric Riemannian manifold the condition  $R(X, Y).P = 0$  hold good.

#### 5. $p$ -Kenmotsu Manifolds Satisfying $P(X, Y).R = 0$

It is known that the condition  $R(X, Y).P = 0$  does not imply  $P(X, Y).R = 0$ . In this section, we study the remarkable property of  $p$ -Kenmotsu manifolds satisfying the condition  $P(X, Y).R = 0$ .

Now, we have

$$\begin{aligned} &(P(X, Y) \cdot R)(U, V)W \\ &= P(X, Y)R(U, V)W - R(P(X, Y)U, V)W \\ &\quad - R(U, P(X, Y)V)W - R(U, V)P(X, Y)W. \end{aligned} \quad (5.1)$$

Put  $X = \xi$  in (5.1). Then the condition  $P(X, Y).R = 0$  implies that

$$\begin{aligned} &g(P(\xi, Y)R(U, V)W, \xi) - g(R(P(\xi, Y)U, V)W, \xi) \\ &- g(R(U, P(\xi, Y)V)W, \xi) - g(R(U, V)P(\xi, Y)W, \xi) \\ &= 0. \end{aligned} \quad (5.2)$$

Putting  $X = \xi, Z = U$  in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned} &\eta(R(P(\xi, Y)U, V)W) \\ &= \eta(U)[\eta(R(Y, V)W) - \frac{1}{(1-n)}\eta(R(QY, V)W)]. \end{aligned} \quad (5.3)$$

Similarly, by putting  $X = \xi, Z = V$  in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned} &\eta(R(U, P(\xi, Y)V)W) \\ &= \eta(V)[\eta(R(U, Y)W) - \frac{1}{(1-n)}\eta(R(U, QY)W)]. \end{aligned} \quad (5.4)$$

In similar by putting  $X = \xi, Z = W$  in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned} &\eta(R(U, V)P(\xi, Y)W) \\ &= \eta(W)[\eta(R(U, V)Y) - \frac{1}{(1-n)}\eta(R(U, V)QY)]. \end{aligned} \quad (5.5)$$

On using (4.3), (5.3), (5.4) and (5.5), we get from eqn (5.2) that

$$\begin{aligned} &\eta(U)[\eta(R(Y, V)W) - \frac{1}{(1-n)}\eta(R(QY, V)W)] \\ &+ \eta(V)[\eta(R(U, Y)W) - \frac{1}{(1-n)}\eta(R(U, QY)W)] \\ &+ \eta(W)[\eta(R(U, V)Y) - \frac{1}{(1-n)}\eta(R(U, V)QY)] = 0. \end{aligned} \quad (5.6)$$

By putting  $Y = U$  in eqn (5.6), we get

$$\begin{aligned} & \eta(U)[\eta(R(U,V)W) - \frac{1}{(1-n)}\eta(R(QU,V)W)] \\ & + \eta(V)[\eta(R(U,U)W) - \frac{1}{(1-n)}\eta(R(U,QU)W)] \quad (5.7) \\ & + \eta(W)[\eta(R(U,V)U) - \frac{1}{(1-n)}\eta(R(U,V)QU)] = 0. \end{aligned}$$

Then on using (2.12) and (2.13), we get

$$\eta(W) \left[ \begin{array}{l} g(U,W)\eta(V) - g(V,W)\eta(U) \\ -\frac{1}{(1-n)}[S(U,U)\eta(V) - S(V,U)\eta(U)] \end{array} \right] = 0. \quad (5.8)$$

Now putting  $U = e_i$ , where  $i = 1, 2, \dots, n$  and taking the summation of (5.8) over  $i$ ,  $1 \leq i \leq n$ , we get  $r = n(n-1)$ , since  $\eta(V) \neq 0$ , shows that the scalar curvature is constant.

Hence we have the following theorem.

**Theorem 5.1:** If a  $p$ -Kenmotsu manifold satisfies the condition  $P(X,Y).R = 0$  then its scalar curvature is constant.

## Acknowledgement

The authors acknowledge Prof. Kalpana, Banaras Hindu University and Dr. B. Satyanarayana of Nagarjuna University for their valuable suggestions in preparation of the manuscript. They are also thankful to the referee for his valuable comments in the improvement of this paper.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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