The Standard Representation of the Symmetric Group S_n over the Ring of Integers

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Abstract In this paper we give a Casimir Invariant for the Symmetric group S_n. Furthermore we obtain and present, for the first time in the literature, explicit formulas for the matrices of the standard representation in terms of the matrices of the permutation representation.

Keywords: standard representation, symmetric group, casimir invariant

1. Introduction

The symmetric group S_n, whose elements consist of the set of all permutations on n symbols is of central importance to mathematics and physics [1]. Cayley's theorem states that every group is isomorphic to a subgroup of the symmetric group on that group. In physics the classification of atomic and nuclear states depends essentially on the properties of S_n [1]. The representation theory of the symmetric group is a well studied subject [1,2,3]. The partitions of n or equivalently Young diagrams of size n are the natural ways in which to parametrize the irreducible representations of S_n [4]. This paper is concerned, not with the general irreducible representations of S_n, but, more specifically, with the so-called standard representation of the symmetric group, formally obtained from the n-1 dimensional subspace of vectors whose sum of coordinates is zero in the basis set of a permutation representation. The path taken in this work shall however be non-group theoretic. For example, we will not be concerned with Young diagrams.

The standard representation of S_n is important for the following reason: For $n \geq 7$, the permutation representation, the trivial (identical) representation, the sign representation, the standard representation and another n-1 dimensional irreducible representation found by tensoring with the sign representation are the only lowest-dimensional irreducible representations of S_n [4]. All other irreducible representations have dimension at least n. While it is a fact that all irreducible representations of S_n can be found, using Frobenius formula ([1], pp 189), for example, there are no known explicit formulas for the standard representation. The main result of this paper is the derivation of such formulas, which now make it possible to write down the standard representation matrices directly from those of the permutation representation.

2. The Permutation Representation

Denote the $n!$ elements of S_n by $A^k, k = 1, 2, ..., n!$, such that, in usual notation,

$$A^k = \begin{pmatrix} 1 & 2 & 3 & 4 & ... & ... & n \\ a_1^k & a_1^k & a_1^k & a_1^k & ... & ... & a_1^k \end{pmatrix}. \qquad (1)$$

and $1 \leq a_i^k \leq n$, , all a_i^k being distinct. For simplicity, and since no ambiguity can result, we will use the same symbol A^k for the representation matrices. Then, in the permutation representation, the $n \times n$ matrices A^k are given, through their elements, by:

$$\left(A^k \right)_{ij} = \delta_{a_i^k, j}, \qquad i = 1...n, \; j = 1...n, \qquad (2)$$

which is clearly a unitary representation, since

$$\left(\left(A^k \right)^\dagger A^k \right)_{ij} = \sum_{r=1}^{n} \left(\left(A^k \right)^\dagger \right)_{ir} A_{ij}^k$$

$$= \sum_{r=1}^{n} A_{ri}^k A_{rj}^k = \sum_{r=1}^{n} \delta_{a_r^k, i} \delta_{a_r^k, j} = \delta_{i,j}.$$

2.1. A Casimir Invariant for S_n in the Permutation Representation

Theorem 1. *The $n \times n$ matrix C, with elements $C_{ij} = 1 - \delta_{ij}$ is a Casimir Invariant of S_n in the permutation representation.*

Proof We require to prove that C commutes with every A^k.

$$\left(CA^k\right)_{ij} = \sum_{r=1}^{n} C_{ir} \left(A^k\right)_{rj} = \sum_{r=1}^{n} (1-\delta_{ir}) \left(A^k\right)_{rj}$$

$$= \sum_{r=1}^{n} \left(A^k\right)_{rj} - \sum_{r=1}^{n} \delta_{ir} \left(A^k\right)_{rj} \quad (3)$$

$$= \sum_{r=1}^{n} \left(A^k\right)_{rj} - \left(A^k\right)_{ij}$$

$$= 1 - \left(A^k\right)_{ij}.$$

A similar calculation gives,

$$\left(A^k C\right)_{ij} = \sum_{r=1}^{n} \left(A^k\right)_{ir} C_{rj}$$

$$= \sum_{r=1}^{n} \left(A^k\right)_{ir} - \left(A^k\right)_{ij} = 1 - \left(A^k\right)_{ij}. \quad (4)$$

We see, therefore, that $A^k C = CA^k$, so that C is a Casimir Invariant of the symmetric group.

3. The Standard Representation

Since the Casimir invariant C, obtained in the previous section, is not proportional to the identity, Schur's lemma tells us that the permutation representation is not irreducible, a well-known fact. It therefore remains to find the matrix P which diagonalizes C. First we prove a lemma.

Lemma 1 *The nonsingular $n \times n$ matrix P with elements $P_{ij} = \delta_{j1} - (1-\delta_{j1})\delta_{i1} + \delta_{i,n+2-j}$ has the inverse P^{-1} where $n\left(P^{-1}\right)_{ij} = 2\delta_{i1} - 1 + n\delta_{i,n+2-j}$*

Proof

$$n\left(PP^{-1}\right)_{ij} = \sum_{r=1}^{n} P_{ir} \, n\left(P^{-1}\right)_{rj}$$

$$= \sum_{r=1}^{n} \begin{pmatrix} \delta_{r1} - \delta_{i1} \\ +\delta_{r1}\delta_{i1} + n\delta_{i,n+2-r} \end{pmatrix} \times \begin{pmatrix} 2\delta_{r1} - 1 \\ +n\delta_{r,n+2-j} \end{pmatrix}. \quad (5)$$

It is straightforward to write out the terms and evaluate the summation termwise. One merely needs to note that

$$\sum_{r=1}^{n} \delta_{r,n+2-j} = 1 - \delta_{j1}$$

and

$$\sum_{r=1}^{n} \delta_{i,n+2-r} \delta_{r,n+2-j} = \delta_{ij} - \delta_{i1}\delta_{j1}.$$

One then finds $\left(PP^{-1}\right)_{ij} = \delta_{ij}$, which establishes the claim.

3.1. Diagonal form of the Casimir Invariant

Theorem 2. *The matrix P given in Lemma 1, diagonalizes the Casimir Invariant, C.*

Proof. We wish to compute

$$D = P^{-1} C P.$$

Now

$$D_{ij} = \sum_{r=1}^{n} \sum_{s=1}^{n} \left(P^{-1}\right)_{ir} C_{rs} P_{sj}.$$

Substituting the matrix elements, expanding and evaluating the sums, we find after some algebra, that

$$D_{ij} = n\delta_{i1}\delta_{j1} - \delta_{ij}. \quad (6)$$

Thus we see that D is a diagonal matrix, as claimed, with the entry '$n-1$' in row 1, column 1 and the remaining diagonal elements being -1.

The matrix P, above, which diagonalizes C will block-diagonalize the matrices A^k.

3.2. Similarity Transformation of A^k: The Standard Representation

Using the matrix elements of P, P^{-1} and A^k, it is not difficult to obtain the interesting result:

$$\left(P^{-1} A^k P\right)_{ij} = \sum_{r=1}^{n} \sum_{s=1}^{n} \left(P^{-1}\right)_{ir} A^k_{rs} P_{sj}$$

$$= \delta_{j1}\delta_{i1} + (1-\delta_{i1})(1-\delta_{j1}) \left[A^k_{n+2-i,n+2-j} - A^k_{n+2-i,1} \right]. \quad (7)$$

We see from (7) that each matrix $P^{-1} A^k P$ is block diagonal, being the direct sum of a 1×1 matrix with entry 1 and an $(n-1) \times (n-1)$ matrix B^k, with elements

$$B^k_{ij} = A^k_{n+1-i,n+1-j} - A^k_{n+1-i,1}, \quad (8)$$
$$i = 1, ..., n-1, \quad j = 1, ..., n-1.$$

The 1×1 matrices correspond to the identical (trivial) representation in which every element of S_n is sent to the one-dimensional identity matrix, while the B^k matrices correspond to the irreducible $n-1$ dimensional standard representation.

4. Conclusion

In this paper we have shown that the operator C with matrix elements $C_{ij} = 1 - \delta_{ij}$ is a Casimir Invariant for the symmetric group S_n. We also showed that if A^k, $k = 1, ..., n!$ are the representation matrices for the elements of S_n in the permutation representation, then the matrices B^k for the standard representation of S_n are given by

$$B^k_{ij} = A^k_{n+1-i,n+1-j} - A^k_{n+1-i,1}, i = 1, ..., n-1, \quad j = 1, ..., n-1.$$

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