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Diagonal Function of k-Lucas Polynomials

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Abstract The Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, Diagonal function of k-Lucas Polynomials is introduced and defined by $G_{n+1}(x) = kxG_n(x) + G_{n-2},(x), n \ge 1.$ with $G_0(x) = 2.$ and $G_1(x) = 1.$ Some Lucas Polynomials, rising & descending diagonal function and generating matrix established and derived by standard methods.

Keywords: *Lucas Polynomials, rising diagonal function, descending diagonal function and generating matrix*

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1. Introduction

The sequence 1,1,2,3,5..., got its name as name as Fibonacci sequence by the Famous Mathematics Francois Edouard Lucas in 1876 [7].

Lucas also discovered a new Fibonacci like sequence with different initial condition call it, Lucas Sequence

$$L_n = L_{n-1} + L_{n-2}, n \ge 2.$$

with initial condition $L_0 = 2, L_1 = 1.$

In 1965 Hoggatt, V.E. [5] has defined Lucas polynomials by recurrence relation. $L_{n+1}(x) = xL_n(x) + L_{n-1},(x),$ where

$$L_0(x) = 2, L_1(x) = x, \quad (1.1)$$

The first few Lucas polynomials are

$$L_1(x) = 1.x^1,$$
$$L_2(x) = 1.x^2 + 2.x^0,$$
$$L_3(x) = 1.x^3 + 3.x^1,$$
$$L_4(x) = 1.x^4 + 4.x^2 + 2.x^0,$$
$$L_5(x) = 1.x^5 + 5.x^2 + 5.x^1,$$
$$L_6(x) = 1.x^6 + 6.x^4 + 9.x^2 + 2.x^0,$$
$$L_7(x) = 1.x^7 + 7.x^5 + 4.x^5 + 7.x^1, ...$$

In this paper, we are using the pair of sequence $\{G_n\}$ and $\{P_n\}$ for which,

$$G_{n+1}(x) = kxG_n(x) + G_{n-1},(x), n \ge 1. \quad (1.2)$$

$$G_0 = 2, G_1 = 1 \quad (x \ne 0)$$

$$P_{n+1}(x) = kxP_n(x) + P_{n-1},(x), n \ge 1$$

$$P_0 = k, P_1 = kx. \quad (x \ne 0) \quad (1.3)$$

where k is any positive integer. k= 0, 1, 2, 3...

Using the equation (1.1) and (1.2) and made a rising diagonal function and descending Diagonal Functions.

2. Sequence $\{G_n\}$ and $\{P_n\}$

We have the pair of sequence $\{G_n\}$ and $\{P_n\}$ for which,

$$G_{n+1}(x) = kxG_n(x) + G_{n-1},(x), n \ge 1.$$
$$G_0 = 2, G_1 = 1 \quad (x \ne 0),$$
$$P_{n+1}(x) = kxP_n(x) + P_{n-1},(x), n \ge 1.$$
$$P_0 = k, P_1 = kx. \quad (x \ne 0)$$

The first few terms of the sequence $\{G_n\}$ are

$$\begin{aligned}
&2\\
&1\\
&kx+2\\
&k^2x^2+2kx+1\\
&k^3x^3+2k^2x^2+2kx+2\\
&k^4x^4+2k^3x^3+3k^2x^2+4kx+1\\
&k^5x^5+2k^4x^4+4k^3x^3+6k^2x^2+3kx+2\\
&k^6x^6+2k^5x^5+5k^4x^4+8k^3x^3+6k^2x^2+6kx+1\\
&k^7x^7+2k^6x^6+6k^5x^5+10k^4x^4\\
&\quad+10k^3x^3+12k^2x^2+4kx+2\\
&k^8x^8+2k^7x^7+7k^6x^6+12k^5x^5+15k^4x^4\\
&\quad+20k^3x^3+10k^2x^2+8kx+1.
\end{aligned} \quad (2.1)$$

The first few terms of the sequence $\{P_n\}$ are

$$\begin{aligned}
& k \\
& kx \\
& k^2x^2+k \\
& k^3x^3+k^2x \;\; +kx \\
& k^4x^4+k^3x^2+2k^2x^2+k \\
& k^5x^5+k^4x^3+3k^3x^3+2k^2x \;\; +kx, \\
& k^6x^6+k^5x^4+4k^4x^4+3k^3x^2+3k^2x^2+k \\
& k^7x^7+k^6x^5+5k^5x^5+4k^4x^3 \\
& \quad +6k^3x^3+3k^2x \;\; +kx \\
& k^8x^8+k^7x^6+6k^6x^6+5k^5x^4 \\
& \quad +10k^4x^4+6k^3x^2+4k^2x^2+k.
\end{aligned} \tag{2.2}$$

3. Rising Diagonal Function

Consider the rising diagonal function of x, $U_n(x)$, $u_n(x)$ for (2.1) and (2.2) respectively,

$$\begin{aligned}
& U_1(x)=1 \\
& U_2(x)=kx \\
& U_3(x)=k^2x^2+2 \\
& U_4(x)=k^3x^3+2kx \\
& U_5(x)=k^4x^4+2k^2x^2+1 \\
& U_6(x)=k^5x^5+2k^3x^3+2kx \\
& U_7(x)=k^6x^6+2k^4x^4+3k^2x^2+2 \\
& U_8(x)=k^7x^7+2k^5x^5+4k^3x^3+4kx \\
& U_9(x)=k^8x^8+2k^6x^6+5k^4x^4+6k^2x^2+1.
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
& u_1(x)=k \\
& u_2(x)=kx \\
& u_3(x)=k^2x^2 \\
& u_4(x)=k^3x^3+k \\
& u_5(x)=k^4x^4+k^2x \\
& u_6(x)=k^5x^5+k^3x^3+kx \\
& u_7(x)=k^6x^6+k^4x^3+2k^2x^2 \\
& u_8(x)=k^7x^7+k^5x^4+3k^3x^3+k \\
& u_9(x)=k^8x^8+k^6x^5+4k^4x^4+2k^2x^2
\end{aligned} \tag{3.2}$$

Now, we define

$$U_0(x)=u_0(x)=0. \tag{3.3}$$

from equation (3.1), (3.2) and (3.3) we get the following theorem:

Theorem (1). If U_n (x) and $u_n(x)$ are rising diagonal functions of x for sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for, $n \ge 4$

$$U_n(x)=kxU_{n-1}(x)+U_{n-4}(x). \tag{3.4}$$

Proof can be obtained by PMI's method so it is obvious.

Special Case-I

If U_n (x) and $u_n(x)$ are rising diagonal functions of x f sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for n=3, n=4.

$$u_n(x)=kxu_{n-1}(x)+u_{n-3}(x). \tag{3.5}$$

4. Descending Diagonal Function

From (2.1) and (2.2), the descending diagonal function of x, $Q_i(x)$, $q_i(x)$ are

$$\begin{aligned}
& Q_1(x)=1 \\
& Q_2(x)=kx+1 \\
& Q_3(x)=(kx+1)^2 \\
& Q_4(x)=(kx+1)^3 \\
& Q_5(x)=(kx+1)^4 \\
& Q_6(x)=(kx+1)^5 \\
& Q_7(x)=(kx+1)^6 \\
& Q_8(x)=(kx+1)^7.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
& q_1(x)=k \\
& q_2(x)=kx+k \\
& q_3(x)=(kx+k)(kx+1) \\
& q_4(x)=(kx+k)(kx+1)^2 \\
& q_5(x)=(kx+k)(kx+1)^3 \\
& q_6(x)=(kx+k)(kx+1)^4 \\
& q_7(x)=(kx+k)(kx+1)^5 \\
& q_8(x)=(kx+k)(kx+1)^6
\end{aligned} \tag{4.2}$$

Now, we define

$$Q_0(x)=q_0(x)=0. \tag{4.3}$$

from (4.1), (4.2) and (4.3) we get for $n \ge 2$.

$$Q_n(x)=(kx+1)Q_{n-1}=(kx+1)^{n-1}. \tag{4.4}$$

$$q_n(x)=(kx+1)q_{n-1}. \tag{4.5}$$

from (4.4) and (4.5) we get the following theorem:

Theorem (2). If $Q_n(x)$ and $q_n(x)$ are descending diagonal function of x for Sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for $n > 2$.

a) $\frac{Q_n}{Q_{n-1}}=\frac{q_n}{q_{n-1}}=(kx+1).$

b) $\frac{Q_n}{q_n}=\frac{(kx+1)}{(kx+k)}.$

5. Generating Matrix

For the sequence $\{G_n\}$ defend in equation (1.1) we consider the matrix

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}. \qquad (5.1)$$

Since, the elements of this matrix are the member of the sequence of Fibonacci Polynomials. We call this matrix as Fibonacci matrix.

Theorem (3). For sequence $\{G_n\}$ we define $n \geq 1$, $p \geq 0$.

$$G_{n+p}(x) = G_{n+1}(x)G_p(x) + G_n(x)G_{p-1}(x)$$
$$G_{n+p}(x) = G_n(x)G_{p+1}(x) + G_{n-1}(x)G_p(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

Since, determinant of matrix A is -1, therefore,

$$\det A^n = (\det A)^n \qquad (5.2)$$

$$\det A^n = (-1)^n$$

$$A^n = \begin{bmatrix} G_{n+1}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix} \qquad (5.3)$$

Form equation (3.5.2) and (3.5.3), we get

$$G_{n+1}(x)G_{n-1}(x) - G_n^2(x) = (-1)^n$$

Since $A^{n+p} = A^n A^p$

$$\begin{bmatrix} G_{n+p+1}(x) & G_{n+p}(x) \\ G_{n+p}(x) & G_{n+p-1}(x) \end{bmatrix}$$
$$= \begin{bmatrix} G_{n+1}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix} \begin{bmatrix} G_{p+1}(x) & G_p(x) \\ G_p(x) & G_{p-1}(x) \end{bmatrix}$$

After multiplying the matrices and equating the corresponding elements, we get

$$G_{n+p}(x) = G_{n+1}(x)G_p(x) + G_n(x)G_{p-1}(x)$$
$$G_{n+p}(x) = G_n(x)G_{p+1}(x) + G_{n-1}(x)G_p(x).$$

Theorem (4). For sequence $\{G_n\}$ we define $n \geq 1$, $p \geq 0$

$$G_n(x) = G_{n+p+1}(x)G_{-n}(x) + G_{n+p}(x)G_{-(p+1)}(x)$$
$$G_n(x) = G_{n+p}(x)G_{-p+1}(x) + G_{n+p-1}(x)G_{-p}(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

If A is any square Matrix, then we know that

$$AA^{-1} = I \qquad (5.4)$$

Where I is identity matrix from equation (5.4) we get

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & -kx \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} G_0(x) & G_{-1}(x) \\ G_{-1}(x) & G_{-2}(x) \end{bmatrix}.$$

By Mathematical induction, we have

$$A^{-p} = \begin{bmatrix} G_{-(p-1)}(x) & G_{-p}(x) \\ G_{-p}(x) & G_{-(p+1)}(x) \end{bmatrix}$$

Since $A^n = A^{n+p}A^{-p}$

$$\begin{bmatrix} G_{n+p}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix}$$
$$= \begin{bmatrix} G_{n+p+1}(x) & G_{n+p}(x) \\ G_{n+p}(x) & G_{n+p-1}(x) \end{bmatrix} \begin{bmatrix} G_{-(p-1)}(x) & G_{-p}(x) \\ G_{-p}(x) & G_{-(p+1)}(x) \end{bmatrix}$$

After multiplying the matrices and equating the corresponding elements, we get

$$G_n(x) = G_{n+p+1}(x)G_{-n}(x) + G_{n+p}(x)G_{-(p+1)}(x)$$
$$G_n(x) = G_{n+p}(x)G_{-p+1}(x) + G_{n+p-1}(x)G_{-p}(x).$$

6. Generating Matrix

For the sequence $\{P_n\}$ defend in equation (1.1) we consider the matrix

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix} \qquad (6.1)$$

since, the elements of this matrix are the members of the sequence of Fibonacci polynomials. We call this matrix as Fibonacci Matrix.

Theorem (5). For sequence $\{P_n\}$ we define $n \geq 1$, $r \geq 0$

$$P_{n+r}(x) = P_{n+1}(x)P_r(x) + P_r(x)P_{r-1}(x)$$
$$P_{n+r}(x) = P_n(x)P_{r+1}(x) + P_{n-1}(x)P_r(x).$$

Proof. For sequence $\{P_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

Since, determinant of matrix A is -1, there for,

$$\det A^n = (\det A)^n \qquad (6.2)$$

$$\det A^n = (-1)^n.$$

By mathematical induction

$$A^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix}. \qquad (6.3)$$

Form equation (3.6.2) and (3.6.3), we get

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n$$

Since $A^{n+r} = A^n A^r$

$$\begin{bmatrix} p_{n+r+1}(x) & p_{n+r}(x) \\ p_{n+r}(x) & p_{n+r-1}(x) \end{bmatrix} = \begin{bmatrix} p_{n+1}(x) & p_n(x) \\ p_n(x) & p_{n-1}(x) \end{bmatrix} \begin{bmatrix} p_{r+1}(x) & p_r(x) \\ p_r(x) & p_{r-1}(x) \end{bmatrix}$$

After multiplying the matrices and equating the corresponding elements, we get

$$\mathrm{P}_{n+r}(x) = \mathrm{P}_{n+1}(x)\mathrm{P}_{r}(x) + P_n(x)P_{r-1}(x).$$

$$\mathrm{P}_{n+r}(x) = \mathrm{P}_{n}(x)\mathrm{P}_{r+1}(x) + P_{n-1}(x)P_{r}(x).$$

Theorem (6). For sequence $\{P_n\}$ we define $n \geq 1$, $r \geq 0$.

$$\mathrm{P}_{n}(x) = \mathrm{P}_{n+r+1}(x)\mathrm{P}_{-r}(x) + P_{n+r}(x)P_{-(r+1)}(x).$$

$$\mathrm{P}_{n}(x) = \mathrm{P}_{n+r}(x)\mathrm{P}_{-r+1}(x) + P_{n+r-1}(x)P_{-r}(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

If A is any square Matrix, then we know that

$$AA^{-1} = I \qquad (6.4)$$

Where I is identity matrix from equation (5.4) we get

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -kx \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} P_n(x) & P_{-1}(x) \\ P_{-1}(x) & P_{-2}(x) \end{bmatrix}$$

By mathematical indication, we have

$$A^{-r} = \begin{bmatrix} P_{-(r-1)}(x) & P_{-r}(x) \\ P_{-r}(x) & P_{-(r+1)}(x) \end{bmatrix}$$

Since $A^n = A^{n+r}A^{-r}$

$$\begin{bmatrix} p_{n+1}(x) & p_n(x) \\ p_n(x) & p_{n-1}(x) \end{bmatrix} = \begin{bmatrix} p_{n+r+1}(x) & p_{n+r}(x) \\ p_{n+r}(x) & p_{n+r-1}(x) \end{bmatrix} \begin{bmatrix} p_{-(r-1)}(x) & p_{-r}(x) \\ p_{-r}(x) & p_{-(r+1)}(x) \end{bmatrix}.$$

After multiplying the matrices and equating the corresponding elements, we get

$$\mathrm{P}_{n}(x) = \mathrm{P}_{n+r+1}(x)\mathrm{P}_{-r}(x) + P_{n+r}(x)P_{-(r+1)}(x).$$

$$\mathrm{P}_{n}(x) = \mathrm{P}_{n+r}(x)\mathrm{P}_{-r+1}(x) + P_{n+r-1}(x)P_{-r}(x).$$

7. Conclusions

In this paper Diagonal function k-Lucas Polynomials. Some basic rising diagonal function and descending diagonal function and generating matrix derived by standard method.

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