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On the Increasing Monotonicity of a Sequence Originating from Computation of the Probability of Intersecting between a Plane Couple and a Convex Body

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Abstract In the paper, the authors confirm the increasing monotonicity of a sequence which originates from computation of the probability of intersecting between a plane couple and a convex body.

Keywords: *increasing monotonicity, sequence, gamma function, ratio of two gamma functions, inequality, logarithmically completely monotonic function, probability of intersecting between a plane couple and a convex body*

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1. Introduction

On 19 December 2014, Mr. Yan-Zong Zhang, a mathematician in China, asked online a question: is the sequence

$$p_n=\frac{n-1}{2}\left(\int_0^{\pi/2}\sin^{n-1}\theta d\theta\right)^2,\quad n\in\mathbb{N} \tag{1.1}$$

increasing? if not, can one take an example? where $\mathbb{N}$ denotes the set of all positive integers. On 20 December 2014, he told that this problem is needed by his teacher, Ms. Jun Jiang, and she said that this problem originates from computation of the probability of intersecting between a plane couple and a convex body in an unpublished paper.

The main of this paper is to give an affirmative answer to the above question.

Theorem 1.1. *The sequence p_n defined by (1.1) is strictly increasing.*

2. A Direct Proof of Theorem 1.1

We firstly affirm the above question directly.

It is well known that the formula

$$\int_0^{\pi/2}\sin^n xdx=\int_0^{\pi/2}\cos^n xdx=\frac{\sqrt{\pi}\Gamma((n+1)/2)}{n\Gamma(n/2)},\ n\in\mathbb{N}$$

is called the Wallis sine (cosine) formula, see [[9], Section 1.1.3], where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\qquad \Re(z)>0$$

is the classical Euler gamma function.

It is clear that $p_1=0$ and $p_n>0$ for $n>1$.

One may reformulate the sequence p_n for $n>1$ in terms of the Euler gamma function $\Gamma(x)$ as

$$q_n=\frac{n}{2}\left(\int_0^{\pi/2}\sin^n\theta d\theta\right)^2=\frac{n}{2}\left[\frac{\sqrt{\pi}\Gamma((n+1)/2)}{n\Gamma(n/2)}\right]^2,$$

for $n\in\mathbb{N}$. Hence, in order to make sure the increasing monotonicity of the sequences p_n and q_n, it is sufficient to make clear the monotonicity property of the sequence

$$Q_n=\frac{1}{n}\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^2,\ n\in\mathbb{N}. \tag{2.1}$$

Taking the logarithm of Q_n gives

$$\ln Q_n=2\left[\ln\Gamma\left(\frac{n+1}{2}\right)-\ln\Gamma\left(\frac{n}{2}\right)\right]-\ln n\triangleq G_n$$

and using the functional equation $\Gamma(x+1)=x\Gamma(x)$ leads to

$$
\begin{aligned}
&G_{n+1}-G_n \\
&=2\left[\ln\Gamma\left(\frac{n+2}{2}\right)+\ln\Gamma\left(\frac{n}{2}\right)-2\ln\Gamma\left(\frac{n+1}{2}\right)\right]+\ln\frac{n}{n+1} \\
&=2\left[\ln\frac{n}{2}+2\ln\Gamma\left(\frac{n}{2}\right)-2\ln\Gamma\left(\frac{n+1}{2}\right)\right]+\ln\frac{n}{n+1} \\
&=4\left[\ln\Gamma\left(\frac{n}{2}\right)-\ln\Gamma\left(\frac{n+1}{2}\right)\right]+\ln\left[\frac{n}{n+1}\left(\frac{n}{2}\right)^2\right] \\
&=\ln\left[\frac{n}{n+1}\left(\frac{n}{2}\right)^2\right]-2(G_n+\ln n) \\
&=\ln\frac{n}{4(n+1)}-2G_n.
\end{aligned}
$$

As a result, it suffices to prove $G_{n+1}-G_n>0$ which is equivalent to

$$
G_n=\ln\left\{\frac{1}{n}\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^2\right\}<\frac{1}{2}\ln\frac{n}{4(n+1)},
$$

that is,

$$
\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^2<\sqrt{\frac{n^3}{4(n+1)}},\quad n\in\mathbb{N}. \tag{2.2}
$$

In [[7], p. 645], Gurland obtained that

$$
\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^2<\frac{n^2}{2n+1},\quad n\in\mathbb{N}. \tag{2.3}
$$

Later, Chu recovered the ineuqlaity (2.3) in [[4], Theorem 2]. Since

$$
\frac{n^2}{2n+1}<\sqrt{\frac{n^3}{4(n+1)}}
$$

is equivalent to

$$
\left(\frac{n^2}{2n+1}\right)^2<\frac{n^3}{4(n+1)},\quad -\frac{n^3}{4(n+1)(2n+1)^2}<0,
$$

the inequality (2.2) is valid. This implies that the sequence Q_n, and then the sequence p_n, is strictly increasing. The proof of Theorem 1.1 is complete.

3. The First Indirect Proof of Theorem 1.1

Now we are in a position to give the first indirect proof of Theorem 1.1.

One may observe that the sequence Q_n defined by (2.1) may be rearranged as

$$
Q_n=\frac{1}{n}\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right]^{1/\left(\frac{n+1}{2}-\frac{n}{2}\right)},\quad n\in\mathbb{N}
$$

and

$$
Q_n=\frac{1}{n}F_{1/2,0}\left(\frac{n}{2}\right), \tag{3.1}
$$

where

$$
F_{a,b}(x)=\frac{1}{x}\left[\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]^{1/(a-b)},\quad x>0. \tag{3.2}
$$

Recall from [1,11] that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)}\ge 0$ for all $k\in\mathbb{N}$ on I. For more information about the notion "logarithmically completely monotonic function", please refer to [2,5,13,16,17,18] and closely related references therein.

In 1986, J. Bustoz and M. E. H. Ismail revealed in essence in [3] that

(1) the function

$$
f_c(x)=\frac{1}{(x+c)^{1/2}}\cdot\frac{\Gamma(x+1)}{\Gamma(x+1/2)},
$$

for $x>\max\left\{-\frac{1}{2},-c\right\}$ is logarithmically completely monotonic on $(-c,\infty)$ if $c\le\frac{1}{4}$, so is the reciprocal of the function $f_c(x)$ on $[-\frac{1}{2},\infty)$ if $c\ge\frac{1}{2}$;

(2) the function

$$
f_{a,b;c}(x)=(x+c)^{a-b}\frac{\Gamma(x+b)}{\Gamma(x+a)}
$$

for $1\ge b-a>0$ is logarithmically completely monotonic on the interval $(\max\{-a,-c\},\infty)$ if $c\le\frac{a+b-1}{2}$, so is the reciprocal of the function $f_{a,b;c}(x)$ on $(\max\{-b,-c\},\infty)$ if $c\ge a$.

The logarithmically complete monotonicity of the function $f_{1/2}(x)$ and $f_{0,1/2;0}(x)$ imply the strictly increasing monotonicity of the function $[F_{1/2,0}(x)]^{1/2}$ on $(0,\infty)$. Therefore, by the relation (3.1), the sequence Q_n, and then the sequence p_n, is strictly increasing. The proof of Theorem 1.1 is complete.

4. The Second Indirect Proof of Theorem 1.1

Finally we give the second indirect proof of Theorem 1.1.

For real numbers a, b, and c, denote $\rho=\min\{a,b,c\}$ and let

$$
H_{a,b,c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)},\quad x\in(-\rho,\infty).
$$

In [[12], Theorem 1], Qi and Guo discovered the following necessary and sufficient conditions:

(1) the function $H_{a,b,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if and only if

$$\begin{aligned}(a,b,c)&\in D_1(a,b,c)\\ &\triangleq\{(a,b,c):(b-a)(1-a-b+2c)\ge 0\}\\ &\cap\{(a,b,c):(b-a)\big(|a-b|-a-b+2c\big)\ge 0\}\\ &\setminus\{(a,b,c):a=c+1=b+1\}\\ &\setminus\{(a,b,c):b=c+1=a+1\};\end{aligned}$$

(2) the function $H_{b,a,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if and only if

$$\begin{aligned}(a,b,c)&\in D_2(a,b,c)\\ &\triangleq\{(a,b,c):(b-a)(1-a-b+2c)\le 0\}\\ &\cap\{(a,b,c):(b-a)\big(|a-b|-a-b+2c\big)\le 0\}\\ &\setminus\{(a,b,c):b=c+1=a+1\}\\ &\setminus\{(a,b,c):a=c+1=b+1\}.\end{aligned}$$

This means that the function

$$H_{1/2,0;0}(x)=\frac{1}{H_{0,1/2;0}}=F_{1/2,0}(x)$$

is strictly increasing on $(0,\infty)$, where $F_{1/2,0}(x)$ is defined by (3.2). As a result, by the relation (3.1), the sequence Q_n, and so the sequence p_n, is strictly increasing. The proof of Theorem 1.1 is complete.

Remark 4.1. The reciprocal of the sequence p_n for $n\ge 2$ is a (logarithmically) completely monotonic sequence. For information on the definition of (logarithmically) completely monotonic sequences and related properties, please refer to closely related chapters in the books [8,19].

Remark 4.2. By carefully reading the expository and survey articles [9,10,14,15] and a large amount of references therein, one may deeply understand and extensively comprehend the spirit and essence of this paper.

Remark 4.3. This paper is a slightly revised version of the preprint [6].

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