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Some Common Fixed Point Theorems for Weakly Contractive Maps in G-Metric Spaces

Manoj Kumar*

Department of Mathematics Guru Jambheshwar University of Science and Technology, Hisar, India
*Corresponding author: manojantil18@gmail.com

Abstract In this paper, first we prove a common fixed point theorem for a pair of weakly compatible maps under weak contractive condition. Secondly, we prove common fixed point theorems for weakly compatible mappings along with E.A. and (CLRf) properties.

Keywords: *weakly compatible maps, weak contraction, generalized weak contraction, altering distance functions, E.A. property, (CLRf) property*

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1. Introduction

In 2006, Mustafa and Sims [6] introduced a new notion of generalized metric space called G-metric space. In fact, Mustafa et. al. [5-9] studied many fixed point results for a self-mapping in G-metric space under certain conditions.

In the present work, we study some fixed point results for a pair of self mappings in a complete G-metric space X under weakly contractive conditions related to altering distance functions.

In 1984, Khan et. al. [4] introduced the notion of altering distance function as follows:

Definition 1.1. A mapping f: $[0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

f is continuous and non-decreasing.

$f(t) = 0 \Leftrightarrow t = 0$.

Definition 1.2. Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all x, y in X, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z in X with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all x, y, z, a in X, (rectangular inequality).

Then the function G is called a generalized metric, or specially a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.3. Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, then $\{x_n\}$ is said to be G-convergent to x in X, if $G(x, x_n, x_m) \to 0$, as $n, m \to \infty$.

G-Cauchy sequence in X, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Proposition 1.4. Let (X, G) be a G-metric space. Then, the following are equivalent

$\{x_n\}$ is G-convergent to x.

$G(x_n, x_n, x) \to 0$, as $n \to \infty$.

$G(x_n, x, x) \to 0$, as $n \to \infty$.

$G(x_n, x_m, x) \to 0$, as $n, m \to \infty$.

Proposition 1.5. Let (X, G) be a G-metric space. Then, the following are equivalent the sequence $\{x_n\}$ is G-Cauchy.

for any $\varepsilon > 0$ there exists k in $\mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq k$.

Proposition 1.6. Let (X, G) be a G-metric space. Then $f : X \to X$ is G-continuous at x in X if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to f(x).

Proposition 1.7. Let (X, G) be a G-metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.8. A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

In 1996, Jungck [3] introduced the concept of weakly compatible maps as follows:

Definition 1.9. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri et. al. [1] introduced the notion of E.A. property as follows:

Definition 1.10. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

In 2011, Sintunavarat et. al. [10] introduced the notion of (CLR_f) property as follows:

Definition 1.11. Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some x in X.

In 2011, Aydi H. [2] introduced the concept of weak contraction in G-metric space as follows:

Definition 1.12. Let (X, G) be a G-metric space. A mapping $f : X \to X$ is said to be a φ-weak contraction, if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$G(fx, fy, fz) \le G(x, y, z) - \varphi(G(x, y, z))$, for all x, y, z in X.

In 2011, Aydi H. [2] proved the following result:

Theorem 1.13. Let X be a complete G-metric space. Suppose the map $f : X \to X$ satisfies the following:

$\psi(G(fx, fy, fz)) \le \psi(G(x, y, z)) - \varphi(G(x, y, z))$, for all x, y, z in X,

where ψ and φ are altering distance functions.

Then f has a unique fixed point (say u) and f is G-continuous at u.

2. Weakly Compatible Maps

Theorem 2.1. Let (X, G) be a G-metric space and let f and g be self mappings on X satisfying the followings:

$$gX \subset fX \tag{2.1}$$

$$fX \text{ or } gX \text{ is complete subspace of } X, \tag{2.2}$$

$$\begin{aligned} &\psi\big(G(gx, gy, gz)\big) \\ &\le \psi\big(G(fx, fy, fz)\big) - \varphi(G(fx, fy, fz)), \\ &\text{where and are altering distance functions.} \end{aligned} \tag{2.3}$$

Then, f and g have a point of coincidence in X.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (2.1), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_{n+1} = gx_n$, n = 0, 1, 2,…

From (2.3), we have

$$\begin{aligned} &\psi\big(G(y_n, y_{n+1}, y_{n+1})\big) = \psi\big(G(gx_n, gx_{n+1}, gx_{n+1})\big) \\ &\le \psi\big(G(fx_n, fx_{n+1}, fx_{n+1})\big) - \varphi\big(G(fx_n, fx_{n+1}, fx_{n+1})\big) \\ &= \psi\big(G(y_{n-1}, y_n, y_n)\big) - \varphi\big(G(y_{n-1}, y_n, y_n)\big) \\ &< \psi\big(G(y_{n-1}, y_n, y_n)\big). \end{aligned} \tag{2.4}$$

Since ψ is non-decreasing, therefore we have

$$G(y_n, y_{n+1}, y_{n+1}) \le G(y_{n-1}, y_n, y_n).$$

Let $u_n = G(y_n, y_{n+1}, y_{n+1})$, then $0 \le u_n \le u_{n-1}$ for all $n > 0$.

It follows that the sequence $\{u_n\}$ is monotonically decreasing and bounded below. So, there exists some $r \ge 0$ such that

$$\lim_{n\to\infty} G(y_{n+1}, y_n, y_n) = \lim_{n\to\infty} u_n = r. \tag{2.5}$$

From (2.4) and (2.5) and letting $n \to \infty$, we have $\psi(r) \le \psi(r) - \varphi(r)$, since ψ and φ are continuous.

Thus, we get $\varphi(r) = 0$, i.e., r = 0, by property of φ, we have

$$\lim_{n\to\infty} G(y_{n+1}, y_n, y_n) = \lim_{n\to\infty} u_n = 0. \tag{2.6}$$

Now, we prove that $\{y_n\}$ is a G-Cauchy sequence. Let, if possible, $\{y_n\}$ is not a G-Cauchy sequence. Then, there exists $\varepsilon > 0$, for which, we can find subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) > k$ such that

$$G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big) \ge \varepsilon \tag{2.7}$$

Let m(k) be the least positive integer exceeding n(k) satisfying (3.7) such that

$$\begin{aligned} &G\big(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\big) < \varepsilon, \\ &\text{for every integer } k. \end{aligned} \tag{2.8}$$

Then, we have

$$\begin{aligned} &\varepsilon \le G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big) \\ &\le G\big(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\big) + G\big(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\big) \\ &< \varepsilon + G\big(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\big). \end{aligned} \tag{2.9}$$

Letting $k \to \infty$, and using (2.6), we have

$$\lim_{k\to\infty} G\big(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\big) = 0.$$

From (2.8), we get

$$\lim_{k\to\infty} G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big) = \varepsilon. \tag{2.10}$$

Moreover, we have

$$\begin{aligned} &G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big) \\ &\le G\big(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\big) + G\big(y_{n(k)-1}, y_{m(k)-1}\ y_{m(k)-1}\big) \\ &+ G\big(y_{m(k)-1}, y_{m(k)}, y_{m(k)}\big), \end{aligned}$$

$$\begin{aligned} &G\big(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\big) \\ &\le G\big(y_{n(k)-1}, y_{n(k)}, y_{n(k)}\big) + G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big) \\ &+ G\big(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}\big). \end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and using (2.6) – (2.10), we get

$$\lim_{k\to\infty} G\big(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\big) = \varepsilon. \tag{2.11}$$

Taking $x = x_{n(k)}$, $y = x_{m(k)}$ and $z = x_{m(k)}$ in (2.3), we get

$$\begin{aligned} &\psi\Big(G\big(y_{n(k)}, y_{m(k)}, y_{m(k)}\big)\Big) \\ &= \psi\Big(G\big(gx_{n(k)}, gx_{m(k)}, gx_{m(k)}\big)\Big) \\ &\le \psi\Big(G\big(fx_{n(k)}, fx_{m(k)}, fx_{m(k)}\big)\Big) \\ &- \varphi\Big(G\big(fx_{n(k)}, fx_{m(k)}, fx_{m(k)}\big)\Big) \\ &= \psi\Big(G\big(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\big)\Big) \\ &- \varphi\Big(G\big(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\big)\Big) \end{aligned}$$

Letting $k \to \infty$, using (2.11) and the continuity of ψ and φ, we get

$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon)$, that is, $\varphi(\varepsilon) = 0$, a contradiction, since $\varepsilon > 0$.

Thus $\{y_n\}$ is a G-Cauchy sequence.

Since fX is complete subspace of X, so there exists a point u $\in$ fX, such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = u. \tag{2.12}$$

Now, we show that u is the common fixed point of f and g.

Since u $\in$ fX, so there exists a point p $\in$ X, such that, fp = u.

From (2.3), we have

$$\psi(G(fp, gp, gp)) = \lim_{n\to\infty} \psi(G(gx_n, gp, gp))$$
$$\le \lim_{n\to\infty} \psi(G(fx_n, fp, fp)) - \lim_{n\to\infty} \varphi(G(fx_n, fp, fp)).$$

Using (2.12) and the property of ψ and φ, we have

ψ (G(fp, gp, gp)) $\le$ ψ (0) – φ (0), implies that, G(fp, gp, gp) = 0, that is, fp = gp = u.

Hence u is the coincidence point of f and g.

Since, fp = gp, and f, g are weakly compatible, we have fu = fgp = gfp = gu.

Now, we claim that, fu = gu = u.

Let, if possible, gu ≠ u.

From (2.3), we have

$$\psi(G(gu, u, u)) = \psi(G(gu, gp, gp))$$
$$\le \psi(G(fu, fp, fp)) - \varphi(G(fu, fp, fp))$$
$$= \psi(G(gu, u, u)) - \varphi(G(gu, u, u))$$
$$< \psi(G(gu, u, u)), \textit{ a contradiction.}$$

Hence gu = u = fu, so u is the common fixed point of f and g.

For the uniqueness, let v be another common fixed point of f and g so that fv = gv = v.

We claim that u = v. Let, if possible, u ≠ v.

From (2.3), we have

$$\psi(G(u, v, v)) = \psi(G(gu, gv, gv))$$
$$\le \psi(G(fu, fv, fv)) - \varphi(G(fu, fv, fv))$$
$$= \psi(G(u, v, v)) - \varphi(G(u, v, v))$$
$$< \psi(G(u, v, v)), \textit{ a contradiction.}$$

Thus, we get, u = v.

Hence u is the common fixed point of f and g.

3. E.A. Property

Theorem 3.1. Let (X, G) be a G-metric space. Let f and g be weakly compatible self maps of X satisfying (2.3) and the followings:

$$f \text{ and } g \text{ satisfy the E.A. property}, \tag{3.1}$$

$$fX \text{ is closed subset of } X. \tag{3.2}$$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = x_0$ for some x_0 in X.

Now, fX is closed subset of X, therefore, by (3.1), we have $\lim_{n\to\infty} fx_n$ = fz, for some z in X.

From (2.3), we have

$$\psi(G(gx_n, gz, gz)) \le \psi(G(fx_n, fz, fz))$$
$$-\varphi(G(fx_n, fz, fz))$$

Letting limit as n → ∞, we have

$$\lim_{n\to\infty} \psi(G(gx_n, gz, gz)) \le \lim_{n\to\infty} \psi(G(fx_n, fz, fz))$$
$$- \lim_{n\to\infty} \varphi(G(fx_n, fz, fz)).$$

Using (2.3), and property of ψ, φ, we have

ψ (G(fz, gz, gz)) ≤ ψ (0) – φ (0) = 0, implies that, G(fz, gz, gz) = 0, that is, fz = gz.

Now, we show that gz is the common fixed point of f and g.

Suppose that gz ≠ ggz. Since f and g are weakly compatible, gfz = fgz and therefore ffa = gga.

From (2.3), we have

$$\psi(G(gz, ggz, ggz))$$
$$\le \psi(G(fz, fgz, fgz)) - \varphi(G(fz, fgz, fgz))$$
$$= \psi(G(gz, ggz, ggz)) - \varphi(G(gz, ggz, ggz))$$
$$< \psi(G(gz, ggz, ggz)), \textit{ a contradiction.}$$

Hence ggz = gz, so gz is the common fixed point of f and g.

Finally, we show that the fixed point is unique.

Let u and v be two common fixed points of f and g such that u ≠ v.

From (2.3), we have

$$\psi(G(u, v, v)) = \psi(G(gu, gv, gv))$$
$$\le \psi(G(fu, fv, fv)) - \varphi(G(fu, fv, fv))$$
$$= \psi(G(u, v, v)) - \varphi(G(u, v, v))$$
$$< \psi(G(u, v, v)), \textit{ a contradiction.}$$

Thus, we get, u = v.

Hence u is the unique common fixed point of f and g.

4. (CLR_f) Property

Theorem 4.1. Let (X, G) be a G-metric space. Let f and g be weakly compatible self maps of X satisfying (2.3) and the following:

$$f \text{ and } g \text{ satisfy } (CLR_f) \text{ property}. \tag{4.1}$$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLR_f) property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$ = fx for some x in X.

From (2.3), we have

$$\psi\left(G\left(gx_n,\ gx,\ gx\right)\right)$$
$$\le \psi\left(G\left(\ fx_n,\ fx,\ fx\right)\right) - \varphi\left(G\left(\ fx_n,\ fx,\ fx\right)\right).$$

Letting limit as n → ∞, we have

$$\lim_{n\to\infty} \psi\left(G\left(gx_n,\ gx,\ gx\right)\right)$$
$$\le \lim_{n\to\infty} \psi\left(G\left(\ fx_n,\ fx,\ fx\right)\right) - \lim_{n\to\infty} \varphi\left(G\left(\ fx_n,\ fx,\ fx\right)\right).$$

Using (2.3), and property of ψ , φ , we have

ψ (G(fz, gz, gz)) ≤ ψ (0) – φ (0) = 0, implies that, G(fx, gx, gx) = 0, that is, fx = gx.

Let w = fx =gx.

Since f and g are weakly compatible, gfx = fgx, implies that, fw = fgx = gfx = gw.

Now, we claim that gw = w.

Let, if possible, gw ≠ w.

From (2.3), we have

$$\psi\left(G\left(gw,\ w,\ w\right)\right) = \psi\left(G\left(gw,\ gx,\ gx\right)\right)$$
$$\le \psi\left(G\left(\ fw,\ fx,\ fx\right)\right) - \varphi\left(G\left(\ fw,\ fx,\ fx\right)\right)$$
$$= \psi\left(G\left(gw,\ w,\ w\right)\right) - \varphi\left(G\left(gw,\ w,\ w\right)\right)$$
$$< \psi\left(G\left(gw,\ w,\ w\right)\right),\ \textit{a contradiction.}$$

Hence gw = w = fw, so w is the common fixed point of f and g.

Finally, we show that the fixed point is unique.

Let v be another common fixed point of f and g such that fv = v =gv.

From (2.3), we have

$$\psi\left(G\left(w,\ v,\ v\right)\right) = \psi\left(G\left(gw,\ gv,\ gv\right)\right)$$
$$\le \psi\left(G\left(\ fw,\ fv,\ fv\right)\right) - \varphi\left(G\left(\ fw,\ fv,\ fv\right)\right)$$
$$= \psi\left(G\left(w,\ v,\ v\right)\right) - \varphi\left(G\left(w,\ v,\ v\right)\right)$$
$$< \psi\left(G\left(w,\ v,\ v\right)\right),\ \textit{a contradiction.}$$

Thus, we get, w = v.

Hence w is the unique common fixed point of f and g.

Example 4.2. Let X = [0, 1] and G(x, y, z) =max{|x-y|, |y-z|, |x-z|}, for all x, y, z in X. Clearly (X, G) is a G-metric space.

Let $fx = \frac{1}{4}x$ and $gx = \frac{1}{8}x$ for each x ∈ X. Then

$$gX = [0, \frac{1}{8}][0, \frac{1}{4}] = fX.$$

Without loss of generality, assume that x > y > z.

Then, G(x, y, z) =|x-z|.

Let ψ (t) = 5t and φ (t) = t. Then

$$\psi\left(G\left(gx,\ gy,\ gz\right)\right) = \psi(\frac{1}{8}|x-z|)$$
$$= 5\frac{1}{8}|x-z| = \frac{5}{8}|x-z|.$$
$$\psi\left(G\left(\ fx,\ fy,\ fz\right)\right) = \psi(\frac{1}{4}|x-z|) = \frac{5}{4}|x-z|.$$
$$\varphi\left(G\left(\ fx,\ fy,\ fz\right)\right) = \varphi(\frac{1}{4}|x-z|) = \frac{1}{4}|x-z|.$$

From here, we have

$$\psi\left(G\left(\ fx,\ fy,\ fz\right)\right) - \varphi\left(G\left(\ fx,\ fy,\ fz\right)\right) = |x-z|.$$

So ψ (G(gx, gy, gz)) < ψ (G(fx, fy, fz)) - φ (G(fx, fy, fz)).

From here, we conclude that f, g satisfy the relation (2.3).

Consider the sequence $\{x_n\}$ = { $\frac{1}{n}$ } so that $\lim_{n\to\infty} fx_n = \mathrm{im}_{n\to\infty}\, gx_n = 0 = f\left(0\right)$, hence the pair (f, g) satisfy the (CLR_f) property. Also, f and g are weakly compatible and 0 is the unique common fixed point of f and g.

From here, we also deduce that $\lim_{n\to\infty} fx_n = \mathrm{im}_{n\to\infty}\, gx_n = 0$, where 0 ∈ X, implies that f and g satisfy E.A. property.

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