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Numerical solution of damped forced oscillator problem using Haar wavelets

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Abstract

We present here the numerical solution of damped forced oscillator problem using Haar wavelet and compare the numerical results obtained with some well-known numerical methods such as Runge-Kutta fourth order classical and Taylor Series methods. Numerical results show that the present Haar wavelet method gives more accurate approximations than above said numerical methods.

Keywords: Haar wavelet method; Differential equation; Operational matrix; Damped forced oscillator.

1 Introduction

During the last few decades considerable efforts have been made using wavelet, towards the development of computational methods to solve numerically linear differential equations encountered in various fields of science and engineering. Wavelet analysis is a new branch of applied science. Wavelet methods are applied to find the numerical solution of problems related to science and engineering. In the last recent years, wavelet methods have been attracted the great interest of researchers of physical and mathematical sciences and many research papers were published in these fields. Recently, many researchers have used Haar and Daubechies wavelets to find the numerical solution of differential and integral equations. Before, the discovery of Haar wavelet, Daubechies wavelets were used in many published research papers for numerical solution of differential and integral equations.

In 1910, Alfred Haar [4] discovered a new wavelet known as Haar wavelet.

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Among all wavelet families, Haar wavelet is most simple, accurate and efficient. It attracted, the interest of many researcher in the field of engineering and science. Haar wavelet has been used in wide variety of numerical methods developed for numerical solutions of differential and integral equations. Here, we present a survey of such methods for differential and integral equations. Chen and Hsiao [3] applied Haar wavelet method for solving lumped and distributed-parameter systems. Hsiao [6] used wavelet approach to time-varying functional differential equations. Razzaghi and Ordokhani [15] used Haar functions for variational problems. Ohkita and Kobayashi [13] applied rationalized Haar functions to solve linear differential equations. Cattani [2] suggested use of Haar wavelet splines for numerical solution of differential equations. Lepik [8, 9, 10, 11, 12] used Haar wavelets for solving differential and integral equations. Sunmonu [18] presented wavelet solution for second order differential equations with maple. Hariharan and Kannan [5] presented an overview of Haar wavelet method for solving differential and integral equations. Kouchi et al. [7] presented numerical solution of homogeneous and inhomogeneous harmonic differential equation with Haar wavelet. In [16], Quasilinearization technique and Haar wavelet operational matrix method both are applied to find the numerical solution of fractional order nonlinear oscillation equations. Also, Solutions of fractional order force-free and forced Duffing-Van der Pol oscillator and higher order fractional Duffing equation on large intervals are presented in [16].

In Section 2, we discussed damped forced oscillator. Haar wavelet method is presented in Section 3. Function approximation is presented in Section 4. In Section 5, we present convergence analysis of Haar wavelet method. In Section 6, the solution by Haar wavelet method is presented. In Section 7, Runge-Kutta method for second order differential equation is presented. Taylor-Series method is presented in Section 8. Comparison of numerical solutions is presented in Section 9 and in Section 10, conclusion is given.

2 Damped forced oscillation

Oscillation means repeated motion of a particle or a body, when displaced from its equilibrium position. The classifications of oscillating systems are presented in Thomsen [19] and in Bhat Rama and Dukkipati [14]. The mechanism that results in dissipation of the energy of an oscillator is called damping. In mechanical oscillator, the damping may be due to (1) Viscous drag (2) Friction and (3) Structure. An oscillator to which a continuous excitation is provided by some external agency is called forced oscillator.

Suppose a mass M attached to the end of a spring of stiffness constant S. One end of the spring is attached to a rigid support. Let the driven force acting on the system be $F(t)$. At any instant, the system will operate under the influence of the following forces:

(a) Restoring force, $F1 = -Sx$ where x is the displacement of the mass from the equilibrium position,
(b) Damping force, $F2 = -rdx/dt$, where r is damping constant,
(c) Driven force, $F3 = F(t)$.
The negative sign in the first two expression implies that both the restoring as well as damping forces opposes the displacement. By Newton second law of motion, we have

$$M\frac{d^2x}{dt^2} = -Sx - r\frac{dx}{dt} + F(t). \tag{1}$$

In this paper, we take special choice $F(t) = 2(1-sint)$, $M = 2kg$, $S = 1N/m$, $r = 0.3Ns/m$ and $x(0) = x'(0) = 0$ as initial conditions, see Simmons [17]. The exact solution of equation (1) by using classical method is:

$$x(t) = e^{-0.075t}(C_1cos(0.703118t) + C_2sin(0.703118t)) + 2 + \frac{200}{109}sin(t) + \frac{60}{109}cos(t). \tag{2}$$

applying initial conditions, we have $C_1 = -\frac{278}{109}$ and $C_2 = -\frac{110425000}{38319931}$.

3 Haar wavelet method

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0, 1]$.

$$h_i(t) = \begin{cases} 1, & \alpha \leq t < \beta, \\ -1, & \beta \leq t < \gamma, \\ 0, & otherwise. \end{cases} \tag{3}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$ and $\gamma = \frac{k+1}{m}$.

Integer $m = 2^j$, $(j = 0, 1, 2, 3, 4, ...J)$ indicates the level of the wavelet. $k = 0, 1, 2, 3, ..., m-1$ is the translation parameter.Maximal level of resolution is J. The index i is calculated according the formula $i = m+k+1$.In the case of minimal values, $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$. where $M = 2^J$. It is assumed that the value $i = 1$, corresponding to the scaling function in $[0, 1]$.

$$h_1(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Let us define the collocation points $t_l = \frac{(l-0.5)}{2M}$, where $l = 1, 2, 3, ..., 2M$ and discredits the Haar function $h_i(t)$.

In the collocation points, the fist four Haar functions can be expressed as follows:
$h_1(t) = [1, 1, 1, 1], h_2(t) = [1, 1, -1, -1], h_3(t) = [1, -1, 0, 0], h_4(t) = [0, 0, 1, -1]$.

We introduce the notation:

$$H_4(t) = [h_1(t), h_2(t), h_3(t), h_4(t)]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{5}$$

Here $H_4(t)$ is called Haar coefficient matrix. It is a square matrix of order 4. The operational matrix of integration P, which is a $2M$ square matrix, is defined by the relations:

$$P_{i,1}(t) = \int_0^{t_l} h_i(t)dt. \tag{6}$$

$$P_{i,n+1}(t) = \int_0^{t_l} P_{i,n}(t)dt, \tag{7}$$

where $n = 1, 2, 3, 4....$

These integrals can be evaluated using equation (3) and first four of them are given below:-

$$P_{i,1}(t) = \begin{cases} t - \alpha, & t\epsilon[\alpha, \beta), \\ \gamma - t, & t\epsilon[\beta, \gamma), \\ 0, & elsewhere, \end{cases} \tag{8}$$

$$P_{i,2}(t) = \begin{cases} \frac{1}{2}(t-\alpha)^2, & t\epsilon[\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - t)^2, & t\epsilon[\beta, \gamma), \\ \frac{1}{4m^2}, & t\epsilon[\gamma, 1), \\ 0, & elsewhere, \end{cases} \tag{9}$$

$$P_{i,3}(t) = \begin{cases} \frac{1}{6}(t-\alpha)^3, & t\epsilon[\alpha, \beta), \\ \frac{1}{4m^2}(t-\beta) - \frac{1}{6}(\gamma - t)^3, & t\epsilon[\beta, \gamma), \\ \frac{1}{4m^2}(t-\beta), & t\epsilon[\gamma, 1), \\ 0, & elsewhere, \end{cases} \tag{10}$$

$$P_{i,4}(t) = \begin{cases} \frac{1}{24}(t-\alpha)^4, & t\epsilon[\alpha, \beta), \\ \frac{1}{8m^2}(t-\beta)^2 - \frac{1}{24}(\gamma - t)^4 + \frac{1}{192m^4}, & t\epsilon[\beta, \gamma), \\ \frac{1}{8m^2}(t-\beta)^2 + \frac{1}{192m^4}, & t\epsilon[\gamma, 1), \\ 0, & elsewhere. \end{cases} \tag{11}$$

4 Function approximation

Any square integrable function $x(t)$ in the interval $[0,1]$ can be expanded by a Haar series of infinite terms:

$$x(t) = \sum_{i=1}^{\infty} a_i h_i(t), i\epsilon\{0\} \cup N \tag{12}$$

where the Haar coefficients a_i are determined as:

$$a_0 = \int_0^1 x(t) h_0(t) dt \tag{13}$$

$$a_n = 2^j \int_0^1 x(t) h_i(t) dt \tag{14}$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$, $x\epsilon[0,1]$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 [x(t) - \sum_{i=0}^{m-1} a_i h_i(t)]^2 dt \tag{15}$$

where $m = 2^j$ and $j\epsilon\{0\} \cup N$.

Usually the series expansion of (12) contains infinite terms for smooth $x(t)$. if $x(t)$ is piecewise constant by itself or may be approximated as piecewise constant during each subinterval, then $x(t)$ will be terminated at finite m terms. This means

$$x(t) = \sum_{i=0}^{m-1} a_i h_i(t) = {a_m}^T h_m(t) \tag{16}$$

where the coefficients ${a_m}^T$ and the Haar function vector $h_m(t)$ are defined as:

$${a_m}^T = [a_0, a_1, a_2, ..., a_{m-1}]$$

and

$$h_m(t) = [h_0(t), h_1(t), h_2(t), ..., h_{m-1}(t)]^T.$$

5 Convergence analysis of Haar wavelet method

Consider a differentiable function $x(t)$ with

$$|x(t)| \leqslant K_0, \tag{17}$$

such that

$$|x'(t)| \leqslant K_0, \tag{18}$$

for all $t\varepsilon(0,1)$. Where $K_0 > 0$ is a positive constant. Haar wavelet approximation for the function $x(t)$ is given by:

$$x_M(t) = \sum_{i=1}^{2M} a_i h_i(t) \tag{19}$$

The square of error norm for wavelet approximation in [1] is given by:

$$\|x(t) - x_M(t)\| \leq \frac{{K_0}^2}{3}.\frac{1}{(2M)^2} \tag{20}$$

Therefore,

$$\|x(t) - x_M(t)\| \leq O(\frac{1}{M}) \tag{21}$$

This means that error bound depends on level of resolution of Haar wavelets that is, error bound is inversely proportional to level of resolution of Haar wavelets. Therefore, when we increase the value of M, it yields the sure convergence of Haar wavelet approximation.

6 Method of solution

Consider the damped forced oscillatory equation (1). Assume that

$$x''(t) = \sum_{i=1}^{2M} a_i h_i(t). \tag{22}$$

Integrating twice with respect to t from 0 to t, we get

$$x'(t) = x'(0) + \sum_{i=1}^{2M} a_i P_{1,i}(t), \tag{23}$$

$$x(t) = x(0) + \sum_{i=1}^{2M} a_i P_{2,i}(t). \tag{24}$$

Apply initial conditions and substitute the values of $x''(t)$, $x'(t)$ and $x(t)$ in (1), we get,

$$\sum_{i=1}^{2M} a_i[Mh_i(t) + rP_{1,i}(t) + SP_{2,i}(t)] = F(t) \tag{25}$$

where r, S, F and M are same as defined in Section 2. From here, wavelet coefficients a_i are calculated and solution $x(t)$ of equation (1) is obtained.

7 Runge-Kutta method of fourth order

Runge-Kutta method is famous numerical method for solving ordinary differential equations. Consider the second order ordinary differential equation

$$\frac{d^2y}{dx^2} = \phi(x, y, \frac{dy}{dx}) \tag{26}$$

By substituting $\frac{dy}{dx} = z$, it can reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z = f(x, y, z) \tag{27}$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \tag{28}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0), \tag{29}$$

$$l_1 = h\phi(x_0, y_0, z_0), \tag{30}$$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1), \tag{31}$$

$$l_2 = h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1), \tag{32}$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2), \tag{33}$$

$$l_3 = h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2), \tag{34}$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3), \tag{35}$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3). \tag{36}$$

Using above relations, we have

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{37}$$

and

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \tag{38}$$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above relations. Similarly by using above relations we compute x_2, y_2, z_2, x_3, y_3, z_3,..........so on.

8 Taylor-series method

Consider equations (26), (27) and (28). If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then, Taylor's algorithm for (26) and (27) gives

$$y_1 = y_0 + h{y_0}' + \frac{h^2}{2!}{y_0}'' + \frac{h^3}{3!}{y_0}''' + ... \tag{39}$$

$$z_1 = z_0 + h{z_0}' + \frac{h^2}{2!}{z_0}'' + \frac{h^3}{3!}{z_0}''' + ... \tag{40}$$

Differentiating (26) and (27) successively, we get y'', z'', etc. So the values ${y_0}', {y_0}'', {y_0}''', ...$ and ${z_0}', {z_0}'', {z_0}''', ...$ are known. Substituting these values in above equations, we get y_1, z_1. Similarly, we have the algorithms

$$y_2 = y_1 + h{y_1}' + \frac{h^2}{2!}{y_1}'' + \frac{h^3}{3!}{y_1}''' + ... \tag{41}$$

$$z_2 = z_1 + h{z_1}' + \frac{h^2}{2!}{z_1}'' + \frac{h^3}{3!}{z_1}''' + ... \tag{42}$$

Since y_1, z_1 are known. we can calculate ${y_1}', {y_1}'', {y_1}''', ...$ and ${z_1}', {z_1}'', {z_1}''', ...$ Substituting these values in above equations, we get y_2, z_2. Proceeding in this way, we can calculate the other values of y and z step by step.

9 Comparison of numerical solutions

In this section, we compare the results of the present Haar wavelet method with two other numerical methods for the damped forced oscillatory problem. In order to verify the efficiency of Haar wavelet method in comparison to exact solution, Runge-kutta fourth order classical method and Taylor series method have been selected. For the Runge-kutta method, the step-size is 1/32. For Taylor's series method, step size is 1/32 and 7 terms are involved. Table-1 shows the numerical results from different numerical methods. Table-2 shows the errors arising from different numerical methods mentioned above. Further, graph in Figure 1 shows the comparison of graphical solution with the exact solution, obtained for $J = 3$ by (i) Haar wavelet method (ii) Runge-Kutta fourth order classical method and (iii) Taylor series method.

Table 1: Results from different numerical methods

$x(l)/32$	Exact solution	Haar wavelet	Runge-Kutta	Taylor series
1	0.0004824193	0.0004707036	0.0004630874	0.0004671912
3	0.0042356457	0.0042007228	0.0040510208	0.0040991982
5	0.0114690029	0.0114111581	0.0109524412	0.0110920238
7	0.0218950125	0.0218146369	0.0208783000	0.0211608344
9	0.0352250351	0.0351226225	0.0335339697	0.0340203049
11	0.0511707741	0.0510469174	0.0486211919	0.0493861034
13	0.0694457517	0.0693011385	0.0658400073	0.0669763459
15	0.0897667509	0.0896021596	0.0848906621	0.0865130140
17	0.1118552180	0.1116715126	0.1054754806	0.1077233303
19	0.1354386196	0.1352367450	0.1273006949	0.1303410858
21	0.1602517482	0.1600327242	0.1500782254	0.1541079134
23	0.1860379714	0.1858028875	0.1732274013	0.1787745029
25	0.2125504200	0.2123004292	0.1973766160	0.2041017526
27	0.2395531092	0.2392894218	0.2213649076	0.2298618522
29	0.2668219898	0.2665458670	0.2452434581	0.2558392936
31	0.2941459241	0.2938586718	0.2687770046	0.2818318053

10 Conclusion

Here, we used three numerical methods to approximate the solutions of damped forced oscillatory differential equation, and compared the results with exact solution. From above results, it is concluded that Haar wavelet method is simple, accurate and more efficient than other well known numerical methods for damped forced oscillatory differential equation.

References

1. Babolian, E. and Shahsawaran, A. *Numerical solution of non-linear fredholm integral equations of the second kind using haar wavelets*, J. Comput. Appl. Math. 225 (2009) 87-95.

2. Cattani, C. *Haar wavelet splines*, J. Interdisciplinary Math. 4 (2001) 35-47.

Table 2: Errors from different numerical methods

$x(l)/32$	Haar wavelet	Runge-Kutta	Taylor series
1	0.0000117157	0.0000193319	0.0000152281
3	0.0000349228	0.0001846249	0.0001364475
5	0.0000578447	0.0005165617	0.0003769791
7	0.0000803755	0.0010167125	0.0007341781
9	0.0001024125	0.0016910654	0.0012047302
11	0.0001238567	0.0025495822	0.0017846707
13	0.0001446131	0.0036057444	0.0024694058
15	0.0001645913	0.0048760888	0.0032537369
17	0.0001837053	0.0063797374	0.0041318877
19	0.0002018746	0.0081379247	0.0050975338
21	0.0002190239	0.0101735228	0.0061438348
23	0.0002350839	0.0128105701	0.0072634685
25	0.0002499908	0.0151738040	0.0084486674
27	0.0002636874	0.0181882016	0.0096912570
29	0.0002761227	0.0215785317	0.0109826962
31	0.0002872523	0.0253689195	0.0123141188

3. Chen, C.F. and Hsiao, C.H. *Haar wavelet method for solving lumped and distributed-parameter systems*, IEEE Proc.: Part D, 144(1) (1997) 87-94.

4. Haar, A. *Zur theorie der orthogonalen Funktionsysteme*, Math. Annal. 69 (1910) 331-371.

5. Hariharan, G. and Kannan, K. *An overview of Haar wavelet method for solving differential and integral equation*, World Applied Sciences Journal 23(12) (2013) 1-14.

6. Hsiao, C.H. *Wavelet approach to time-varying functional differential equations*, Int. J. Computer Math. 87(3) (2008) 528-540.

7. Kouchi, M.R., Khosravi, M. and Bahmani, J. *A numerical solution of Homogengous and Inhomogeneous Harmonic Differential equation with Haar wavelet*, Int. J. Contemp. Math. Sciences 6(41) (2011) 2009-2018.

8. Lepik, Ü. *Numerical solution of differential equations using Haar wavelets*, Math. Comput. Simulat. 68 (2005) 127-143.

9. Lepik, Ü. *Application of Haar wavelet transform to solving integral and differential equation*, Proc. Estonian Acad. Sci. Phys. Math. 56(1) (2007) 28-46.

10. Lepik, Ü. *Haar wavelet method for higher order differential equations*, Int. J. MAth. Comput. 1 (2008) 84-94.

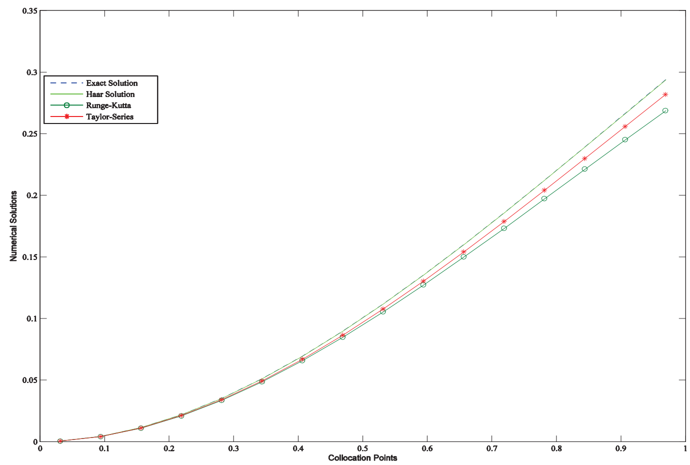

Figure 1: Comparison of graphical solution of different numerical methods.

11. Lepik, Ü. *Haar wavelet method for solving stiff differential equations*, Math. Modeling and Analysis 4 (2009) 467-489.

12. Lepik, Ü. *Solution of optimal control problems via Haar wavelets*, Int. J. Pure. Appl. Math. 55 (2009) 81-94.

13. Ohkita, M. and Kobayashi, Y. *An application of rationalized Haar functions to solution of linear differential equations*, IEEE Trans. Circuit System 9 (2003) 853-862.

14. Rama, B.B. and Dukkipati, V. *Advanced dynamics*, Alpha Science, Pangbourne, 2001.

15. Razzaghi, M. and Ordokhani,Y. *An application of rationalized Haar functions for variational problems*, Appl. MAth. Comput. 122 (2001) 353-364.

16. Saeed, U. and Rehman, M.U. *Haar wavelet operational matrix method for fractional oscillation equations*, International Journal of Mathematics and Mathematical Sciences, (2014) 1-8.

17. Simmons, G.F. *Differential equations with applications and historical notes*, Mcgraw-Hill, London. 1972.

18. Sunmonu, A. *Implementation of wavelet solution to second order differential equations with maple*, Applied Mathematical Sciences 6(127) (2012) 6311-6326.

19. Thomsen, J.J. *Vibrations and stability order and chaos*, Mc-graw-Hill, London, 1997.

حل عددی مساله نیروی میرای نوسانی با استفاده از موجکهای هار

ایندریپ سینک و شیوکومار

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چکیده : در این مقاله، حل عددی مساله نیروی میرای نوسانی را با استفاده از موجکهای هار، ارائه می کنیم. نتایج عددی حاصل را با نتایج حاصل از به کار بردن بعضی از روشهای مشهور، مانند رانگ–کوتای مرتبه چهارم کلاسیک و روشهای سری تیلور مقایسه می کنیم.

نتایج عددی نشان می دهند که روش استفاده از موجک های هار، که در این مقاله ارائه گردیده است، جوابهای تقریبی دقیق تری را در مقایسه با روش های ذکر شده در بالا، بدست می دهند.

کلمات کلیدی : روش موجک های هار؛ معادلات دیفرانسیل؛ ماتریس عملیاتی؛ نیروی میرای نوسانی.