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FINITELY DUAL QUASI-REGULAR RELATIONS

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Abstract. In article [**4**], quasi-regular relations and dually quasi-regular relations on sets are introduced and analyzed. In this paper, following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations on sets, the concept of finitely dual quasi-regular relations is introduced, as a continuation of [**4**]. Characterizations of finitely dual quasi-regular relations are obtained.

1. Introduction

The concept of conjugative relations was introduced by Guanghao Jiang and Luoshan Xu ([**1**]), and the concept of dually normal relations was introduced and analyzed by Jiang Guanghao and Xu Luoshan in [**2**]. In article [**4**], this author introduced and analyzed a new class of relations on set - the class of *quasi-regular relations* on sets. In this article, as a continuation of article [**4**], following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations on sets, introduced in articles [**1**] and [**2**], we introduce and analyze notion of finitely dual quasi-regular relations on sets

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. In this semigroup the relation $\triangle_X = \{(x, x) : x \in X\}$ is the unity. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \ \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

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It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$. Also, if $A = \{a_1, a_2, ..., a_k\}$, then $A\alpha = \bigcup_{i\in\{1,2,...,k\}} a_i\alpha$.

Undefined notions and notations used in this paper the reader can find in the book [**3**]

2. Dually quasi-regular relations

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:

Definition 2.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is:
– *dually normal* ([**2**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([**1**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

The notion of dually quasi-regular relation on a set is introduced by this author in his article [**4**].

Definition 2.2. For relation $\alpha \in \mathcal{B}(X)$ we say that it is *dually quasi-regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^C.$$

The family of dually quasi-regular relations is not empty. Let $\alpha \in \mathcal{B}(X)$ be a relation such that $(\alpha^C)^{-1} \circ \alpha^C = \triangle_X$. Then, we have $\alpha = \alpha \circ \triangle_X = \alpha \circ (\alpha^C)^{-1} \circ \alpha^C$. So, such relation α is a dually quasi-regular relation on X.

In the following proposition we give a characterization of dually quasi-regular relations combining theorem 4.1 and theorem 2.1 of paper [**4**]. Here, for relation α, we use notation α_* to sign the maximal relation of family of relations β such that $\alpha \circ \beta \circ \alpha^C \subseteq \alpha$.

Theorem 2.1. *For a binary relation α on a set X, the following conditions are equivalent:*
(1) *α is a dually quasi-regular relation.*
(2) *For all $x, y \in X$, if $(x, y) \in \alpha$, there exist $u, v \in X$ such that:*
  (a) $(x, u) \in \alpha^C \wedge (v, y) \in \alpha$
  (b) $(\forall s, t \in X)((s, u) \in \alpha^C \wedge (v, t) \in \alpha \Longrightarrow (s, t) \in \alpha)$.
(3) $\alpha \subseteq \alpha \circ \alpha_* \circ \alpha^C$.

3. Finitely dual quasi-regular relations

In this section we introduce the concept of finitely dual quasi-regular relations and give a characterization of that relations. For that we need the concept of *finite extension* of a relation. That notion and belonging notation we borrow from articles [**1**] and [**2**]. For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty }\}$.

Definition 3.1. ([**1**], Definition 3.3; [**2**], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite extension* of α, by

$$(\forall F, G \in X^{(<\omega)})((F,G) \in \alpha^{(<\omega)} \iff G \subseteq F\alpha).$$

From Definition 3.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F,G) \in (\alpha^C)^{(<\omega)} \iff G \subseteq F\alpha^C).$$

Now, we can introduce concept of *finitely dual quasi-regular relation.*

DEFINITION 3.2. A binary relation α on a set X is called *finitely dual quasi-regular* if there exists a relation $\beta^{(<\omega)}$ on $X_{(<\omega)}$ such that

$$\alpha^{(<\omega)} = \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}.$$

Although it is seems, in accordance with Definition 2.2, it would be better to call a relation α on X to be finitely dual quasi-regular if its a finite extension to $X^{(<\omega)}$ is a dually quasi-regular relation, we will not use that option. That concept is different from our concept given by Definition 3.2.

Now we give an essential characterization of finitely dual quasi-regular relations.

THEOREM 3.1. *A binary relation α on a set X is a finitely dual quasi-regular relation if and only if for all $F, G \in X^{(<\omega)}$, if $G \subseteq F\alpha$, then there are $U, V \in X^{(<\omega)}$, such that*
(i) $U \subseteq F\alpha^C$, $G \subseteq V\alpha$, *and*
(ii) *for all $S, T \in X^{(<\omega)}$, if $U \subseteq S\alpha^C$ and $T \subseteq V\alpha$, then $T \subseteq S\alpha$.*

PROOF. (1) Let α be a relation on X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(A,B) \in \beta^{(<\omega)} \iff (\forall S, T \in X^{(<\omega)})((A \subseteq S\alpha^C \wedge T \cap B\alpha \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $\alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, then there are $A, B \in X^{(<\omega)}$ with $(H,A) \in (\alpha^C)^{(<\omega)}$, $(A,B) \in \beta^{(<\omega)}$ and $(B,W) \in \alpha^{(<\omega)}$. Then $A \subseteq H\alpha^C$ and $W \subseteq B\alpha$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $A \subseteq S\alpha^C$ and $B \cap \alpha T \neq \emptyset$ because $w \in B\alpha$. Since $(A,B) \in \beta^{(<\omega)}$, we have that $A \subseteq S\alpha^C \wedge T \cap B\alpha \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H,W) = (S,W) \in \alpha^{(<\omega)}$. Therefore, we have $\alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

The second, check that (b) $\alpha^{(<\omega)} \subseteq \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), then there are $A, B \in X^{(<\omega)}$ such that
(i') $A \subseteq H\alpha^C$, $W \subseteq B\alpha$, and
(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^C$ and $T \subseteq B\alpha$, then $T \subseteq S\alpha$.

Now, we have to show that $(A,B) \in \beta^{(<\omega)}$. Let for all $(C,D) \in (X^{(<\omega)})^2$ be the following $A \subseteq C\alpha^C$ and $D \cap B\alpha \neq \emptyset$ hold. From $D \cap B\alpha \neq \emptyset$ follows that there exists an element $d \in D \cap B\alpha\,(\neq \emptyset)$. So, $d \in D$ and $d \in B\alpha$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^C \wedge T = \{d\} \subseteq B\alpha) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $D \cap S\alpha \neq \emptyset$. Therefore, $(A,B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, from $(H,A) \in (\alpha^C)^{(<\omega)}$, $(A,B) \in \beta^{(<\omega)}$ and $(B,W) \in \alpha^{(<\omega)}$ follows that $(H,W) \in \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$.

By assertions (a) and (b), finally we have $\alpha^{(<\omega)} = \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$.

(2) Let α is a finitely dual quasi-regular relation. Then there is a binary relation $\beta^{(\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $\alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$. So, for all $(F,G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F,G) \in \alpha^{(<\omega)}$, we have $(F,G) \in (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ \alpha^{(<\omega)}$. Whence there is a pair $(U,V) \in (X^{(<\omega)})^2$ such that $(F,U) \in (\alpha^C)^{(<\omega)}$, $(U,V) \in \beta^{(<\omega)}$ and $(V,G) \in \alpha^{(<\omega)}$, i.e., $U \subseteq F\alpha^C$, $G \subseteq V\alpha$. Hence we get the condition (i). Now we check the condition (ii). For all $(S,T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^C$ and $T \subseteq V\alpha$, i.e., $(S,U) \in (\alpha^C)^{(<\omega)}$ and $(V,T) \in \alpha^{(<\omega)}$, then by $(U,V) \in \beta^{(<\omega)}$, we have $(S,T) \in \alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, i.e., $(S,T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$. □

If we put $F = \{x\}$ and $G = \{y\}$ in the previous theorem, we give the following corollary.

Corollary 3.1. *Let α be a relation on set X. Then α is a finitely dual quasi-regular relation in X if and only if for all elements $x, y \in X$ such that $(x,y) \in \alpha$ there are finite non-empty subsets U and V of X such that*
(1^0) $(\forall u \in U)((x,u) \in \alpha^C) \wedge (\exists v \in V)((v,y) \in \alpha)$*; and*
(2^0) *for all $S \in X^{(<\omega)}$ and $t \in X$ holds*

$$(U \subseteq S\alpha^C \wedge t \in V\alpha) \Longrightarrow t \in S\alpha.$$

Proof. (a) Let α be a finitely dual quasi-regular relation in X and let x, y be elements of X such that $(x,y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem 3.1 then there exist finite subsets U and V of X such that conditions (1^0) and (2^0) hold.

(b) Opposite, let $x, y \in X$ such that $(x,y) \in \alpha$ be there are finite non-empty subsets U and V of X such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(A,B) \in \beta^{(<\omega)} \Longleftrightarrow (\forall S \in X^{(<\omega)})(\forall t \in X)((A \subseteq S\alpha^C \wedge t \in B\alpha) \Longrightarrow t \in S\alpha).$$

The proof that the equality $\alpha^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$ holds is the same as in the Theorem 3.1. So, the relation α is a finitely dual quasi-regular. □

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