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ON THE b-CHROMATIC NUMBER OF SOME GRAPHS

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Abstract. The b-coloring of a graph is a variant of proper coloring in which every color class has at least one vertex which has at least one neighbor in every other color class. The largest integer k for which the graph has a b-coloring is called b-chromatic number. We investigate b-chromatic numbers for shell, gear and generalised web graphs.

1. Introduction

A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class c_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring. A b-coloring of a graph G is a proper coloring of G in which each color class has a b-vertex, that is, a vertex that has at least one neighbor in each of the other color class. The concept of b-coloring was introduced in 1999 by Irving and Manlove [**6**]. The b-chromatic number, $\phi(G)$, of G is the largest integer k such that G has a b-coloring using k colors. If G has a b-coloring by k colors for every integer k satisfying $\chi(G) \leqslant k \leqslant \varphi(G)$ then G is called b-continuous. The discussion on the b-chromatic number of some power graphs is carried out by Effantin and Kheddouci [**5**]. The b-continuity property of various graphs is explored by Barth *et al.* [**2**]. The *b*-coloring of regular graphs is studied by Blidia *et al.* [**3**] while the b-coloring of regular graphs without 4 cycle is studied by Shaebani [**9**]. The b-chromatic number for path related graphs is discussed by Vaidya and Rakhimol [**10**]. The same authors have also investigated the b-chromatic numbers of the degree splitting graphs of path, shell and gear graph in [**11**].

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All graphs considered in this paper are finite, simple, connected and undirected. We investigate the b-chromatic numbers of shell, gear and generalised web graphs.

Definition 1.1. ([**6**]) The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m-1$.

Proposition 1.1 ([**4**]). *For any graph G, $\chi(G) \geqslant 3$ if and only if G has an odd cycle.*

Proposition 1.2 ([**7**]). $\chi(G) \leqslant \varphi(G) \leqslant m(G)$.

It is obvious that if $\chi(G) = k$, then every coloring of a graph G by k colors is a b-coloring of G.

Proposition 1.3 ([**1**]). *If C_n, $K_{m,n}$ and $W_n : C_n + K_1$ are respectively cycle, complete bipartite graph and wheel graph, then*

(1) $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$.
(2) $\chi(W_n) = 3$, *if n is even and* $\chi(W_n) = 4$, *if n is odd.*
(3) $\chi(K_{m,n}) = 2$.
(4) $\varphi(W_n) = 3$, *if $n = 4$ and* $\varphi(W_n) = 4$, *if $n \neq 4$.*

Proposition 1.4 ([**4**]). *If G_1 is a subgraph of G_2, then $\chi(G_1) \leqslant \chi(G_2)$.*

2. Main Results

Definition 2.1. ([**8**]) A shell graph S_n is the graph obtained by taking $(n-3)$ concurrent chords in a cycle C_n.

Theorem 2.1. $\phi(S_n) = \begin{cases} 3, & n = 3,4,5 \\ 4, & n \geqslant 6. \end{cases}$

Proof. The graph S_n contains two vertices, say v_1 and v_{n-1}, of degree 2, $n-3$ vertices , say $v_2, v_3, ..., v_{n-2}$, of degree 3 and a vertex, u, of degree $n-1$. Thus $V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\}$ and $|V(S_n)| = n$. As each S_n contains at least an odd cycle C_3, $\phi(S_n) \geqslant \chi(S_n) \geqslant 3$.

For all $n = 3,4,5$, the graphs S_3, S_4 and S_5 has m-degree 3. Thus $\phi(S_n) \leqslant 3$. Consequently $\phi(S_n) = 3$.

But when $n = 6$, the graph S_6 has m-degree 4. Thus $\phi(S_6) \leqslant 4$. If we assign the proper coloring as $c(v_1) = c(v_4) = 1$, $c(v_2) = c(v_5) = 2$, $c(v_3) = 3$ and $c(u) = 4$, we get a b-coloring with b-vertices v_4, v_2, v_3 and u for the color classes $1, 2, 3$ and 4 respectively. Thus $\phi(S_6) = 4$.

The graph S_7 is obtained from S_6 by adding a vertex v_6 and making v_6 adjacent to u and v_5. The addition of a vertex and two edges will not change the number of m-degree vertices of the resultant graph S_7. Consequently, $\phi(S_7) \leqslant 4$. Thus $4 = \phi(S_6) \leqslant \phi(S_7) \leqslant 4$. Therefore $\phi(S_7) = 4$.

By recursive construction of graphs $S_8, S_9, ..., S_n$, each graph S_n has m-degree 4 and repeating the color after assigning the colors used for S_6 we have $\phi(S_n) = 4$. $\square$

Definition 2.2. Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is subdivided when it is replaced by edges $e' = uw$ and $e'' = vw$.

Definition 2.3. The gear Graph, G_n, is obtained from the wheel by subdividing each of its rim edge exactly once.

Lemma 2.1. *G_n is bipartite and $\chi(G_n) = 2$, for all n.*

Proof. Let $W_n = C_n + K_1$ be the wheel graph with apex vertex v and the rim vertices $v_1, v_2, ..., v_n$. To obtain the gear graph G_n, subdivide each rim edge of wheel W_n by the vertices $u_1, u_2, ..., u_n$ where each u_i subdivides the edge $v_i v_{i+1}$ for $i = 1, 2, .., n-1$ and u_n subdivides the edge $v_1 v_n$.

The wheel $W_n = C_n + K_1$ contains an odd cycle C_3 formed by the vertices v_i, v_{i+1} and v. Hence it is not bipartite. But after subdividing each rim edges by the vertices u_i the graph G_n contains only even cycles. Hence it becomes bipartite. Consequently, $\chi(G_n) = 2$, for all n. □

Theorem 2.2. *For all n, $\varphi(G_n) = 4$.*

Proof. Each G_n has at least four vertices of degree 3. Thus $\varphi(G_n) \leqslant m(G_n) = 4$. By assigning proper coloring to the vertices as $c(v) = 4$, $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(u_1) = 3$, $c(u_2) = 1$, $c(u_3) = 2$ and $c(u_n) = 2$ (for the remaining vertices we proceed with any proper coloring) we get the b-vertices v_1, v_2, v_3 and v for the color classes $1, 2, 3$ and 4 respectively. Thus $\varphi(G_n) = 4$. □

Corollary 2.1. *G_3 is not b-continuous.*

Proof. By Lemma 2.1, we have $\chi(G_3) = 2$. Hence b-coloring using 2 colors can be done in G_3. By Theorem 2.2, $\varphi(G_3) = 4$. But, if we use 3 colors to meet the requirement of b- coloring, due to the adjacency of vertices, we cannot have a b-vertex for any one of the three color classes. Hence b-coloring of G_3 is not possible. Thus G_3 is not b- continuous. □

Remark 2.1. As mentioned in M. Alkhateeb ([**1**]) G_3 is the smallest non b-continuous graph and the only one with seven vertices.

Definition 2.4. The web graph is the graph obtained by joining the pendant vertices of a Helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

The graph $W(t, n)$ is the generalised web graph with t number of n- cycles.

Theorem 2.3.

$$\begin{aligned} \varphi(W(t,3)) &= 4, \quad && \textit{when } t = 2, 3 \\ \varphi(W(t,n)) &= 5, \quad && \textit{when } t \neq 2, 3 \textit{ and for all } n \end{aligned}$$

Proof. Consider the generalised web graph $W(t, n)$ with t number of n-cycles. Let $V(W(t, n)) = \{v_i^j, u, u_i; 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant t\}$. $W(t, n)$ has at least 5 vertices of degree at least 4. Therefore $\varphi(W(t, n)) \leqslant 5$. Here $d(u) = n$, $d(u_i) = 1$, $d(v_i^j) = 4$.

Case 1 : When $t = 2$, $n = 3$. Suppose $W(2, 3)$ does have b-chromatic 5 coloring. Then at least two vertices of the first cycle (wlog v_1^1 and v_2^1) are b-vertices. To become b-vertices, v_1^1 and v_2^1 must be adjacent to the three same colors. Such a coloring is not possible since v_1^1 and v_2^1 are adjacent. Therefore $\varphi(W(2, 3)) < 5$.

Thus $\varphi(W(2,3)) \leqslant 4$. As it contains a wheel W_3 and $\chi(W_3) = 4$, $\varphi(W(2,3)) \geqslant \chi((W(2,3)) \geqslant \chi(W_3) = 4$. Hence $\varphi(W(2,3)) = 4$.

Case 2: When $t = 3$, $n = 3$. Suppose $W(3,3)$ does have b-chromatic 5 coloring. By assigning the proper coloring as $c(v_1^1) = 2$, $c(v_2^1) = 5$, $c(v_3^1) = 3$, $c(v_1^2) = 4$, $c(v_2^2) = 3$, $c(v_3^2) = 5$, $c(v_1^3) = 1$, $c(v_2^3) = 5$, $c(v_3^3) = 3$, $c(u) = 1$, $c(u_1) = 2$, $c(u_2) = 3$, $c(u_3) = 2$ gives the b-vertices for the color classes $1, 2$ and 4, but not for 3 and 5. Similarly all other proper coloring using 5 colors will generate b-vertices at most for three color classes only. Hence $\varphi(W(3,3)) \neq 5$. Thus $\varphi(W(3,3)) \leqslant 4$. As it contains a wheel W_3 and $\chi(W_3) = 4$, $\varphi(W(3,3)) \geqslant \chi((W(3,3)) \geqslant \chi(W_3) = 4$. Hence $\varphi(W(3,3)) = 4$.

Case 3: When $t = 4$.

Subcase 1: $n = 3$: If we assign the proper coloring for the graph $W(4,3)$ as $c(v_1^1) = 1$, $c(v_1^2) = 2$, $c(v_1^3) = 5$, $c(v_1^4) = 3$, $c(v_3^1) = 3$, $c(v_3^2) = 4$, $c(v_3^4) = 4$, $c(v_2^1) = 5$, $c(v_2^4) = 1$, $c(v_2^3) = 4$, $c(v_3^3) = 1$, $c(v_2^4) = 1$, $c(u) = 4$ and $c(u_1) = 2$, this gives the b-vertices v_1^1, v_1^2, v_1^4, v_2^3 and v_1^3 for the color classes $1, 2, 3, 4$ and 5 respectively. $\varphi(W(4,3)) = 5$.

Sub case 2: $n \neq 3$: If we assign the proper coloring for the graph $W(4,n)$ as $c(u) = 4$, $c(v_1^1) = 1$, $c(v_1^2) = 2$, $c(v_1^3) = 5$, $c(v_1^4) = 3$, $c(v_2^1) = 5$, $c(v_2^2) = 3$, $c(v_2^3) = 4$, $c(v_2^4) = 2$, $c(v_n^1) = 3$, $c(v_n^2) = 4$, $c(v_n^3) = 1$, $c(v_n^4) = 4$, $c(v_3^1) = 2$, $c(u_1) = 1$, $c(u_2) = 3$, $c(u_n) = 5$ (for the remaining vertices assign proper coloring) we get b-vertices v_1^1, v_1^2, v_1^4, u and v_1^3 for the color classes $1, 2, 3, 4$ and 5 respectively. $\varphi(W(4,n)) = 5$.

Case 4: When $t \geqslant 5$. If we assign the proper coloring for the graph $W(5,n)$ as $c(v_1^1) = 1$, $c(v_1^2) = 2$, $c(v_1^3) = 5$, $c(v_1^4) = 3$, $c(v_1^5) = 4$, $c(u) = 4$, $c(v_2^1) = 5$, $c(v_2^2) = 3$, $c(v_2^3) = 4$, $c(v_2^5) = 2$, $c(v_n^1) = 3$, $c(v_n^2) = 4$, $c(v_n^3) = 1$, $c(v_n^4) = 2$, $c(v_n^5) = 5$, $c(u_1) = 1$ (for the remaining vertices assign proper coloring) we get b-vertices v_1^1, v_1^2, v_1^4, v_1^5 and v_1^3 for the color classes $1, 2, 3, 4$ and 5 respectively. $\varphi(W(5,n)) = 5$.

When $t > 5$, we repeat the coloring as above. It is obvious that $\varphi(W(t,n)) = 5$ as $m(W(t,n)) = 5$. Hence the theorem. □

3. Concluding Remarks

In this paper we investigate b- chromatic numbers of shell, gear and generalised web graphs. The shell and generalised web graphs are obviously b-continuous while the b-continuity of gear graph is discussed in detail. To investigate similar results for other graphs or graph families is an open area of research.

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