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BOUNDARY EDGE DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a connected graph, A subset S of $E(G)$ is called a boundary edge dominating set if every edge of $E - S$ is edge boundary dominated by some edge of S. The minimum taken over all boundary edge dominating sets of a graph G is called the boundary edge domination number of G and is denoted by $\gamma_b'(G)$. In this paper we introduce the edge boundary domination in graph. Exact values of some standard graphs are obtained and some other interesting results are established.

1. Introduction and Definitions

For graph-theoretical terminology and notations not defined here we follow Buck-ley [**6**], West [**8**] and Haynes et al.[**7**]. All graphs in this paper will be finite and undirected, without loops and multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A line graph L(G) (also called an interchange graph or edge graph) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. For terminology and notations not specifically defined here we refer reader to [**11**].

Let G = (V, E) be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. For

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$i \neq j$, a vertex v_i is a boundary vertex of v_j if $d(v_j; v_t) \leqslant d(v_j; v_i)$ for all $v_t \in N(v_i)$ (see [3, 4]).

A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as $N_b(u) = \{v \in V : d(u,w) \leqslant d(u,v) \text{ for all } w \in N(u)\}$. The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(G)$ and $\delta_b(G)$. That is $\Delta_b(G) = \max_{u \in V} |N_b(u)|, \delta_b(G) = \min_{u \in V} |N_b(u)|$.

A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. A subset B of V(G) is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$. [**2**]

The distance $d(e_i, e_j)$ between two edges in $E(G)$ is defined as the distance between the corresponding vertices e_i and e_j in the line graph of G, or if $e_i = uv$ and $e_j = u^{'}v^{'}$, the distance between e_i and e_j in G is defined as follows:

$$d(e_i, e_j) = \min\{d(u,u^{'}), d(u,v^{'}), d(v,v^{'}), d(v,u^{'})\}.$$

The degree of an edge $e = uv$ of G is defined by $deg(e) = degu + degv - 2$.

The concept of edge domination was introduced by Mitchell and Hedetniemi [**12**]. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma^{'}(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. We need the following theorems.

Theorem 1.1. [**2**]

a. *For any path P_n , $n \geqslant 3$, $\gamma_b(P_n) = n - 2$.*

b. *For any complete graph K_n, $n \geqslant 4$, $\gamma_b(K_n) = 1$.*

Theorem 1.2. [**10**] *For any (n,m)-graph G, $\gamma^{'}(G) \leqslant m - \Delta^{'}(G)$, where $\Delta^{'}(G)$ denotes the maximum degree of an edge in G.*

Theorem 1.3. [**5**] *Edge - eccentric graph of a complete graph K_n is a regular graph with regularity $\frac{(n-3)(n-2)}{2}$.*

Theorem 1.4. [**1**] *If $diam(G) \leqslant 2$ and if none of the three graphs F_1, F_2 , and F_3 depicted in Fig. 2 are induced subgraphs of G, then $diam(L(G)) \leqslant 2$.*

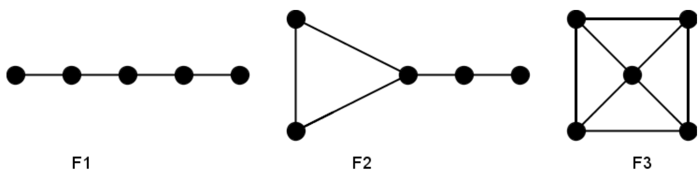

Figure 1. The graphs mentioned in Theorem 1.4

2. Results

Let G = (V,E) be a graph and f, e be any two edges in E. Then f and e are adjacent if they have one end vertex in common.

DEFINITION 2.1. An edge $e = uv \in E$ is said to be a boundary edge of f if $d(e, g) \leqslant d(e, f)$ for all $g \in N^{'}(e)$. An edge g is a boundary neighbor of an edge f if g is a nearest boundary of f. Two edges f and e are boundary adjacent if f adjacent to e and there exist another edge g adjacent to both f and e.

DEFINITION 2.2. A set S of edges is called a boundary edge dominating set if every edge of $E - S$ is boundary edge dominated by some edge of S. The minimum taken over all edge boundary dominating sets of a graph G is called the boundary edge domination number of G and is denoted by $\gamma_b^{'}(G)$.

The boundary edge neighbourhood of e denoted by $N_b^{'}(e)$ is defined as $N_b^{'}(e) = \{f \in E(G) : d(e, g) \leqslant d(e, f)$ for all $g \in N^{'}(e)\}$. The cardinality of $N_b^{'}(e)$ is denoted by $deg_b(e)$ in G. The maximum and minimum boundary degree of an edge in G are denoted respectively by $\Delta_b^{'}(G)$ and $\delta_b^{'}(G)$. That is $\Delta_b^{'}(G) = \max_{e \in E} |N_b^{'}(e)|$ and $\delta_b^{'}(G) = \min_{e \in E} |N_b^{'}(e)|$.

A boundary edge dominating set S is minimal if for any edge $f \in S, S - \{f\}$ is not boundary edge dominating set of G. A subset S of E is called boundary edge independent set, if for any $f \in S, f \notin N_b^{'}(g)$, for all $g \in S - \{f\}$. If an edge $f \in E$ be such that $N_b^{'}(f) = \phi$ then j is in any boundary edge dominating set. Such edges are called boundary-isolated. The minimum boundary edge dominating set denoted by $\gamma_b^{'}(G)$-set.

An edge dominating set X is called an independent boundary edge dominating set if no two edges in X are boundary-adjacent. The independent boundary edge domination number $\gamma_{bi}^{'}(G)$ is the minimum cardinality taken over all independent boundary edge dominating sets of G. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

In Figure 2, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and $E(G) = \{1, 2, 3, 4, 5, 6\}$. The minimum boundary dominating set is $B = \{v_2, v_3\}$. Therefore $\gamma_b(G) = 2$. The minimal edge dominating sets are $\{2, 5\}, \{3, 6\}, \{4, 6\}, \{1, 4\}, \{1, 3, 5\}$. Therefore $\gamma^{'}(G) = 2$.
The minimum boundary edge dominating sets are $\{1, 2, 6\}, \{1, 5, 4\}, \{2, 3, 4\}, \{4, 5, 6\}$. Therefore $\gamma_b^{'}(G) = 3$.
From the definition of line graph and the boundary edge domination the following Proposition is immediate.

OBSERVATION 1. *For any graph G, we have,* $\gamma_b^{'}(G) = \gamma_b(L(G))$

The boundary edge domination number of some standard graphs are given below.

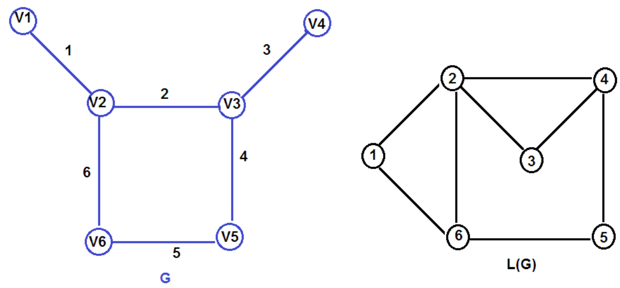

FIGURE 2. G and L(G)

THEOREM 2.1. *For any path P_n , $n \geqslant 3$, $\gamma_b^{'}(P_n) = n - 3$.*

PROOF. Let $G \cong P_n, n \geqslant 3$, by Theorem 1.1(a), we have $\gamma_b(P_n) = n - 2$. since $L(P_n) = P_{n-1}$, then $\gamma_b^{'}(P_n) = \gamma_b(P_{n-1}) = n - 3$. □

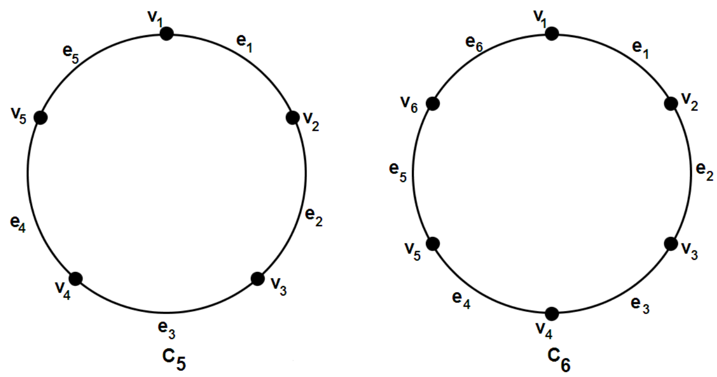

FIGURE 3. C_5 and C_6

EXAMPLE 2.1. In C_5, $\{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}$ are dominating sets and $\{v_1, v_2\}$, $\{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}$ are boundary dominating sets. Therefore $\gamma(C_5) = 2$ and $\gamma_b(C_5) = 2$.
In C_6, $\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}$ are dominating sets and $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$, $\{v_4, v_5\}, \{v_5, v_6\}$ are boundary dominating sets. Therefore $\gamma(C_6) = 2$ and $\gamma_b(C_6) = 2$.

THEOREM 2.2. *For any cycle C_n, $n \geqslant 4$, $\gamma_b(C_n) = \lceil \frac{n}{3} \rceil$.*

PROOF. Every cycle C_n have n vertices and $m = n$ edges in which each vertex is of degree 2. That is each vertex dominates two vertices and in any odd cycle $C_{2m+1}, m \geqslant 2, deg(v) = deg_b(v) = 2$ for all $v \in V(C_{2m+1})$, then we have two cases
Case1: $n \equiv 0\ (mod 3)$.
Let $v_1, v_2, ..., v_n$ be the vertices of C_n. If $v_1 \in B_1$, then $v_2, v_n \notin N_b(v_1)$. It is clear that $d(v_1, v_2) \leqslant d(v_1, v_3)$ and $d(v_1, v_n) \leqslant d(v_1, v_{n-1})$. So $v_3, v_{n-1} \in N_b(v_1)$. If $v_4 \in B_1$, then $v_3, v_5 \notin N_b(v_4)$. and $v_2, v_6 \in N_b(v_4)$.
Similarly we can proceed up to all the n vertices. Finally we get the minimum boundary dominating set is $B_1 = \{v_1, v_4, ..., v_{n-5}, v_{n-2}\}$.
Case 2: $n \equiv 1, 2\ (mod 3)$.
If $v_1 \in B_2$, then $v_2, v_n \notin N_b(v_1)$. it is clear that $d(v_1, v_2) \leqslant d(v_1, v_3)$ and $d(v_1, v_n) \leqslant d(v_1, v_{n-1})$. So $v_3, v_{n-1} \in N_b(v_1)$. If $v_4 \in B_2$, then $v_3, v_5 \notin N_b(v_4)$. and $v_2, v_6 \in N_b(v_4)$. Similarly we can proceed up to all the n vertices. Finally we get the minimum boundary dominating set is $B_2 = \{v_1, v_4, ..., v_{n-3}, v_n\}$.
Therefore the minimum boundary dominating set of C_n is

$$B = \begin{cases} B_1 = \{v_1, v_4, ...v_{n-5}, v_{n-2}\} & \text{if } n \equiv 0\ (mod 3), \\ B_2 = \{v_1, v_4, ..., v_{n-3}, v_n\} & \text{if } n \equiv 1, 2\ (mod 3). \end{cases}$$

$$|B| = \begin{cases} |B_1| = \frac{n-3}{3} + 1 & \text{if } n \equiv 0\ (mod 3), \\ |B_2| = \frac{n-1}{3} + 1 & \text{if } n \equiv 1, 2\ (mod 3). \end{cases}$$

$$= \begin{cases} \frac{n-3}{3} + 1 & \text{if } n \equiv 0\ (mod 3), \\ \frac{n-1}{3} + 1 & \text{if } n \equiv 1, 2\ (mod 3)\ . \end{cases}$$

$$= \begin{cases} \frac{n}{3} & \text{if } n \equiv 0\ (mod 3), \\ \frac{n}{3} + \frac{2}{3} & \text{if } n \equiv 1, 2\ (mod 3). \end{cases}$$

Hence $\gamma_b(C_n) = \lceil \frac{n}{3} \rceil$. □

THEOREM 2.3. *For any cycle C_n, $n \geqslant 4$, $\gamma_b^{'}(C_n) = \lceil \frac{n}{3} \rceil$.*

PROOF. It can be easily observed that $\gamma_b^{'}(C_n) = \lceil \frac{n}{3} \rceil$.
From the definition of line graph we have $L(C_n)$ is C_n , and by Theorem 2.2, then $\gamma_b^{'}(C_n) = \gamma_b(C_n) = \lceil \frac{n}{3} \rceil$. □

THEOREM 2.4. *For a complete graph K_n with $n \geqslant 4$ vertices, $\gamma_b^{'}(K_n) = 3$.*

PROOF. Let $G \cong L(K_n)$, $diam(K_n) = 1$. By Theorem 1.4, $diam(G) \leqslant 2$. If $n = 3$, then $G = K_3$ and $diam(G) = diam(K_3) = 1$.
If $n \geqslant 4$, then G is the strongly regular graph with parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$. which is graph of diameter 2.
suppose $v \in V(G)$, then there exist $\frac{(n-3)(n-2)}{2}$ boundary neighbor vertices of v. and $2(n-2)$ are adjacent vertices to v. If a vertex u is adjacent v then there exist $n-2$ vertices are common between v and u, which contain $n-3$ vertices are boundary neighbors of u . Now we take a vertex w which is adjacent to both v and u , then there exist $n-3$ vertices are boundary neighbors of w and are adjacent to both v and u together.
since

$|N_b(v) \cup N_b(u) \cup N_b(w) \cup \{v,u,w\}| = \frac{(n-3)(n-2)}{2} + 2(n-3) + 3$
$= \frac{n(n-1)}{2} = |V(G)|$.
Therefore the boundary dominating set is $S = \{v,u,w\}$.Which contained any complete K_3 in G. Hence $\gamma_b^{'}(K_n) = 3$. □

THEOREM 2.5. *For a complete bipartite graph $K_{m,n}$, $2 \leqslant m \leqslant n$, $\gamma_b^{'}(K_{m,n}) = m$.*

DEFINITION 2.3. $B_{m,n}$ is the bistar obtained from two disjoint copies of $K_{1,m}$ and $K_{1,n}$ by joining the centre vertices through an edge.

In a bistar there are $m+n+2$ vertices and $m+n+1$ edges, there are totally $m+n$ pendant vertices and 2 centre vertices. The degree of the pendant vertices are 1 and the degree of the central vertices are $m+1$ and $n+1$ obtained from both the m and n edges of $K_{1,m}$ and $K_{1,n}$ respectively and the common edge of the centres [**9**].

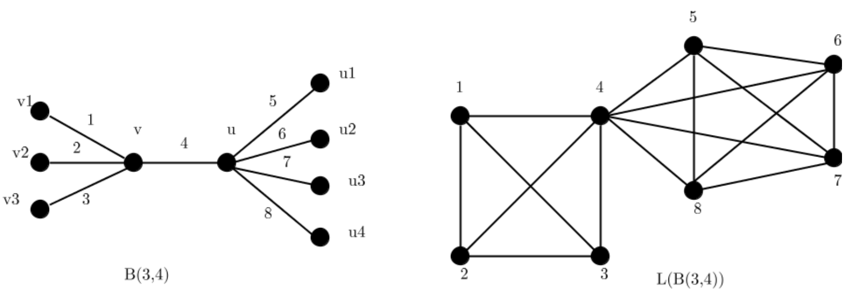

FIGURE 4. $B_{3,4}$ and $L(B_{3,4})$

EXAMPLE 2.2. Let G be the bistar graph $B_{3,4}$ and $L(B_{3,4})$ in the Figure 4, $V(G) = \{v,u,v_1,v_2,v_3,u_1,u_2,u_3,u_4\}$. The minimum boundary dominating set is $B = \{u,v\}$ Therefore $\gamma_b(G) = 2$. The minimum boundary edge dominating set is $\{4\}$. Therefore $\gamma_b^{'}(G) = 1$.

THEOREM 2.6. *For any graph G , $\gamma_b^{'}(G) = 1$ if and only if $G \cong K_{1,n}$ or $B_{m,n}$*

PROOF. Suppose that $\gamma_b^{'}(G) = 1$, Let S denote the set of all boundary edge dominating of G such that $|S| = 1$, we have $\gamma_b^{'}(G) = \gamma_b(L(G))$, then $\gamma_b(L(G)) = 1$. From Theorem 1.1.(b), $\gamma_b(K_n) = 1$, so $L(G) = K_n$. Since the line graph of $K_{1,n}, n \geqslant 3$ is K_n and the line graph of the bistar $B_{m,n}$ is the one point say e union of 2 complete graphs K_m and K_n. Therefor $G \cong K_{1,n}$ or $B_{m,n}$.

Conversely, suppose $G \cong K_{1,n}$ or $B_{m,n}$ the line graph of $K_{1,n}, n \geqslant 3$ is K_n, and the line graph of the bistar $B_{m,n}$ is the one point say e union of 2 complete graphs K_m and K_n. Hence $\gamma_b^{'}(G) = \gamma_b(K_n) = 1$. □

PROPOSITION 2.1. *Let e be an edge of a connected graph G. Then $E - N_b^{'}(e)$ is a boundary edge dominating set for G.*

THEOREM 2.7. *If G is a connected graph of size $m \geqslant 3$, then $\gamma_b^{'}(G) \leqslant m - \Delta_b^{'}(G)$.*

PROOF. Let e be an edge of a connected graph G. Then by the above proposition, $E - N_b^{'}(e)$ is a boundary edge dominating set for G.$(|N_b^{'}(e)| = \Delta_b^{'}(e))$. But $|N_b^{'}(e)| \geqslant 1$. Thus $\gamma_b^{'}(G) \leqslant m - 1$. Suppose $\gamma_b^{'}(G) = m - 1$. Then there exists a unique edge e^* in G such that e^* is a boundary edge neighbour of every edge of $E - \{e^*\}$, this is a contradiction to the fact that in a graph there exist at least two boundary edges. Thus $\gamma_b^{'}(G) \leqslant m - 2$. Hence $\gamma_b^{'}(G) \leqslant m - \Delta_b^{'}(G)$.

□

THEOREM 2.8. *Let G and $\overline{G}$ be connected complementary graphes. Then,*

$\gamma_b^{'}(G) + \gamma_b^{'}(\overline{G}) \leqslant n.$

$\gamma_b^{'}(G) \cdot \gamma_b^{'}(\overline{G}) \leqslant 3(n-3).$

PROOF. If $G \cong L(K_n)$, then G is the strongly regular graph with parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$. which is graph of $\gamma_b^{'}(K_n) = 3$. And $\overline{G}$ also the strongly regular graph of parameters $(\frac{n(n-1)}{2}, \frac{(n-2)(n-3)}{2}, \frac{(n-4)(n-5)}{2}, \frac{(n-3)(n-4)}{2})$. suppose $v \in \overline{G}$ then $deg(v) = \frac{(n-2)(n-3)}{2}$, and there exits $\frac{n(n-1)}{2} - \frac{(n-2)(n-3)}{2}$ are boundary neighbor vertices of v, and there exits $\frac{(n-3)(n-4)}{2}$ are common vertices between v and any vertex u is not adjacent to v. Similarly we can proceed up to all the n vertices. Finally we get a boundary edge domination of $\overline{G}$ is $\gamma_b^{'}(\overline{G}) = \frac{(n-2)(n-3)}{2} - \frac{(n-3)(n-4)}{2} = n - 3$. Hence

$\gamma_b^{'}(G) + \gamma_b^{'}(\overline{G}) \leqslant n.$

$\gamma_b^{'}(G) \cdot \gamma_b^{'}(\overline{G}) \leqslant 3(n-3).$

□

THEOREM 2.9. *For any (n,m)-graph G, $\gamma^{'}(G) + \gamma_b^{'}(G) \leqslant m + 1$.*

PROOF. Let $e \in E$, then $N^{'}(e) \cup N_b^{'}(e) \cup \{e\} = E, |N^{'}(e)| + |N_b^{'}(e)| + 1 = m$ and $\Delta^{'}(G) + \Delta_b^{'}(G) + 1 = m$. But we have $\gamma^{'}(G) \leqslant m - \Delta^{'}(G)$ and $\gamma_b^{'}(G) \leqslant m - \Delta_b^{'}(G)$. Therefor $\gamma^{'}(G) + \gamma_b^{'}(G) \leqslant 2m - (\Delta^{'}(G) + \Delta_b^{'}(G)) = 2m - m + 1 = m + 1$.
Hence $\gamma^{'}(G) + \gamma_b^{'}(G) \leqslant m + 1$. □

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