

Vulnerability of Some Splitting Graphs

Vecdi AYTAÇ and Sema BODUR

ABSTRACT. Vulnerability concerns the issue of network robustness to malicious attacks. Many complex systems in the real world can be conceptually described as networks, where nodes represent the system constituents and edges depict the interaction between them. Often enough, the network representation of these systems is an undirected and unweighted graph which greatly simplifies the structure of complex systems. If we think of a graph as modeling a network, then we have some graph parameters to measure the vulnerability, including connectivity, toughness, binding number, domination number and integrity. In this paper, we consider the *integrity* and *strong domination numbers* as vulnerability measures. We determine the *integrity* and *strong domination numbers* of splitting graphs $S'(G)$ for specific graphs G .

1. Introduction

To model various systems like chemical, social systems, neural networks or the World Wide Web (www) and Internet, networks and complex systems are commonly used. Without any doubt, a very vital part of networks is the network topology which is the center of attention of mathematics and biological, computer and physical sciences. While dealing with networks, the issue of an interconnection network is great importance. It may have different architectural structures which need to be analyzed. For this purpose mathematics usually use graph theory as the most powerful tool. It is known that the underlying topology of an interconnection network is modeled by a graph $G = (V, E)$, where V and E stands for the set of processors and the set of communication links in the network, respectively. While analyzing complex networks, stability, which is a key aspect in designing computer networks, and vulnerability, which can be defined as the measurement of the global power of its related graph, must be taken into account. If graph theoretical

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parameters are used to express the network requirements, the issue of analysis and design of networks becomes finding a graph G which satisfies certain pre-specified requirement. It is known that communication systems are often exposed to failures and attacks. In the literature various measures are suggested to measure the robustness of network and a variety of graph-theoretic parameters have been used to derive formulas to calculate network reliability.

In an analysis of the vulnerability of networks to disruption, three important quantities (there may be others) that come to mind are

- (1) the number of elements that are not functioning,
- (2) the number of remaining connected subnetworks and
- (3) the size of a largest remaining group within which mutual communication can still occur.

Based on these quantities, a number of graph parameters, such as connectivity, toughness, scattering number, integrity, tenacity and their edge-analogues, have been proposed for measuring the vulnerability of networks.

All graphs considered in this paper are finite and undirected, without loops and multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G , denoted by $n = |V(G)|$, is the number of vertices in G . The open neighbourhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. The degree of vertex v of graph G denoted by $\deg(v)$, that is the size of its open neighbourhood [3, 8].

The integrity of graphs is based on above quantities (1) and (3). The integrity of a graph G , $I(G)$, is defined by

$$I(G) = \min \{ |S| + m(G - S) : S \subset V(G) \}$$

where $m(G - S)$ denotes the maximum order of the components of $G - S$. This concept was introduced by Barefoot, Etringer and Swart [1].

The another measures of the graph vulnerability is the domination number. The study of domination in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research. There are several types of domination depending upon the nature of dominating set. In the following, we give the definition strong domination number in connected graphs.

A set $S \subseteq V$ is a dominating set of G , if $N[S] = V$. The domination number $\gamma(G)$ is defined as the minimum cardinality of a dominating set of G [7]. Strong and weak domination was introduced by Sampathkumar and Latha [9]. If $uv \in E$, then u and v dominate each other. Further, u *strongly dominates* v if $\deg(u) \geq \deg(v)$. A set $D \subseteq V$ is strong dominating set (*sd-set*), if every vertex $v \in V - D$ is strongly dominated by some u in D . The strong domination number $\gamma_s(G)$ of G is the minimum cardinality of an *sd-set*. if G is a regular graph, then $\gamma(G) = \gamma_s(G)$ [6, 9, 10]. We use $\lfloor x \rfloor$ to denote the largest integer not greater than x , and $\lceil x \rceil$ to denote the least integer not less than x . We let $x \equiv_l y$ mean $x \equiv y \pmod{l}$.

The paper proceeds as follows. In Section 2, some of the existing literature on

integrity number, domination and strong domination number. The integrity and strong domination numbers splitting graph $S'(G)$ when G is a specified family of graphs are computed in Section 3. Finally, Section 4 concludes the paper.

2. Some of the Results

In this section, we will review some of the well-known theorems about the integrity and strong domination number of some special graphs.

THEOREM 2.1. [1, 2] *The integrity of*

- a) *the complete graph K_n is n ;*
- b) *the null graph \bar{K}_n is 1;*
- c) *the star graph $K_{1,n}$ is 2;*
- d) *the path graph P_n is $\lceil 2\sqrt{n+1} \rceil - 2$;*
- e) *the cycle graph C_n is $\lceil 2\sqrt{n} \rceil - 1$;*
- f) *the comet graph $C_{n-r,r}$ is $I(C_{n-r,r}) = \begin{cases} I(P_n), & \text{if } r \leq \sqrt{n+1} \\ \lceil 2\sqrt{n-r} \rceil - 1, & \text{otherwise} \end{cases}$;*
- g) *the complete bipartite graph $K_{m,n}$ is $1 + \min\{m, n\}$;*

THEOREM 2.2. [5] *Let G be a graph of order n ,*

- a) *$I(G) = 1$ if and only if G is null.*
- b) *$I(G) = 2$ if and only if all nontrivial components of G are edges or the only non-trivial component is a star.*
- c) *$I(G) = n$ if and only if G is complete*

THEOREM 2.3. [4] *The strong domination number of*

- a) *the complete graph K_n is $\gamma_s(K_n) = 1$.*
- b) *the cycle C_n is $\gamma_s(C_n) = \lceil n/3 \rceil$ for $n \geq 3$*
- c) *the path P_n is $\gamma_s(P_n) = \lceil n/3 \rceil$ for $n \geq 2$*
- d) *the wheel W_n is $\gamma_s(W_n) = 1$ for $n \geq 3$*
- e) *the complete bipartite graph $K_{r,t}$ is $\gamma_s(K_{r,t}) = \begin{cases} 2, & \text{if } 2 \leq r = t \\ r, & \text{if } 1 \leq r < t \end{cases}$.*

PROPOSITION 2.1. [9] For a graph G of order n ,

$$\gamma(G) \leq \gamma_s(G) \leq n - \Delta(G)$$

where $\Delta(G)$ is the maximum degree of the graph G .

3. Vulnerability of Some Splitting Graphs

In this section, we consider the integrity, domination and strong domination number of the splitting graphs $S'(G)$ when G is a specified family of graphs. Then, we give the following definition.

DEFINITION 3.1. For a graph the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' , respectively.

Some notations are used in order to make the proof of the given theorems understandable. Let $S'(G)$ be H . If we think that the vertex-set of graph H be where,

$$V(H) = V_1(H) \cup V_2(H)$$

$V_1(H)$: The set contains the vertices of graph G . Let v_1, v_2, \dots, v_n be these vertices.

$V_2(H)$: The set contains new vertices which are obtained by definition of splitting graph. Let u_1, u_2, \dots, u_n be these new added vertices of the graph.

, then it is easily calculated that the integrity, domination and strong domination number of graph H .

3.1. Integrity Value of Some Splitting Graphs.

THEOREM 3.1. If $G = P_n$, then the integrity value of $S'(G) = H$ is

$$I(H) \leq \lceil 4\sqrt{1+n} \rceil - 4$$

PROOF. Graph H contains two P_n subgraphs. In any one of the subgraphs P_n , if r number of vertices are omitted, the remaining one of the components has at least $\frac{n-r}{r+1}$ number of vertices. Considering the above provided expression, from the formation of H graph, if we omit $2r$ number of vertices, the remaining one of the components will have at least $\frac{2n-2r}{r+1}$ number of vertices. If these values are substituted in the integrity formula,

$$I(H) \geq \min_{S \subset V} \{|S| + m(H - S)\}$$

the function $f(r) = 2r + \frac{2n-2r}{r+1}$ will have a minimum value of $r = -1 + \sqrt{1+n}$.

Therefore, the result will be $I(H) \leq 4(\sqrt{1+n} - 1)$. Since the integrity is an integer value, we round this off to get a lower bound,

$$I(H) \leq \lceil 4\sqrt{1+n} \rceil - 4$$

and this completes our proof. \square

THEOREM 3.2. If $G = K_n$, then the integrity value of $S'(G) = H$ is

$$I(H) = n + 1$$

PROOF. The degree of the set of vertices $V_1(H)$ is $(n - 1) + (n - 1) = 2n - 2$. $\forall v \in V_1(H)$ vertex, it will be adjacent to all other vertices in $V_1(H)$ except itself and while forming the graph H , in set $V_2(H)$, apart from its corresponding vertex u , it will be adjacent to the $n - 1$ number of vertices. All vertices in $V_2(H)$ will have a degree of $n - 1$. For $\forall u \in V_2(H)$ vertex, its non adjacent to any other vertex. However, in graph H with set of vertices $V_1(H)$, apart from its corresponding vertex v , it will be adjacent to $V_1(H) - \{v\}$ number of vertices. Let $S \subset V(H)$ and S be a set satisfying the integrity value of graph H . Then we have 3 cases for S .

Case 1. If S is a cover set of H , then $|S| = \alpha(H)$ and $m(H - S) = 1$. By writing these values in the integrity formula from the definition will result to,

$$I(H) = \min_{S \subset V} \{|S| + m(H - S)\}$$

$$I(H) = \alpha(H) + 1$$

Case 2. If $|S| < \alpha(H)$, then, $|S| = \alpha(H) - p$, whereby $p = 1, 2, 3, \dots, \alpha(H) - 1$ and $m(H - S) > p + 1$. If we write these values in the integrity formula from the definition, then we have

$$I(H) = \min_{S \subset V} \{|S| + m(H - S)\}$$

$$I(H) > \min_{S \subset V} \{\alpha(H) - p + p + 1\}$$

$$I(H) > \alpha(H) + 1$$

Case 3. If $|S| > \alpha(H)$, then, $|S| \geq \alpha(H) + 1$ and $m(H - S) = 1$. If we write this values in the integrity formula from the definition, then we have

$$I(H) = \min_{S \subset V} \{|S| + m(H - S)\}$$

$$I(H) \geq \min_{S \subset V} \{\alpha(H) + 1 + 1\}$$

$$I(H) \geq \alpha(H) + 2$$

From the results founded in all three *Case 1, 2, 3*, the integrity value of graph H can be written as

$$I(H) = \alpha(H) + 1$$

Since $\alpha(H) = n$, then $I(H) = n + 1$. □

THEOREM 3.3. *If $G = C_n$, then the integrity value of $S'(G) = H$ is*

$$I(H) \leq \lceil 4\sqrt{n} \rceil - 2$$

PROOF. Graph H contains two C_n subgraphs. In any of the subgraphs C_n , if r number of vertices are omitted, the remaining one of the components will have at least $\frac{n-r}{r}$ number of vertices. If we follow the same procedure of proof like on the Theorem 3. 1. , we have

$$I(H) \leq \lceil 4\sqrt{n} \rceil - 2$$

□

THEOREM 3.4. *If $G = K_{1,n}$, then the integrity value of $S'(G) = H$ is*

$$I(H) = 3$$

PROOF. By taking central vertex $c \in V_1(H)$ and $c' \in V_2(H)$ then omitting $V(H) - \{c, c'\}$, graph H will have isolation vertices and its integrity value when written from the definition formula will be;

$$I(H) = \min_{S \subset V} \{|S| + m(H - S)\}$$

$$I(H) = \min_{S \subset V} \{2 + 1\} = 3$$

□

3.2. Domination and Strong Domination Value.

THEOREM 3.5. *If $G = C_n$, then the strong domination number of $S'(G) = H$ is*

$$\gamma_s(H) = \begin{cases} \lceil \frac{2n}{4} \rceil + 1, & n \equiv_4 2 \\ \lceil \frac{2n}{4} \rceil, & \text{otherwise} \end{cases}$$

PROOF. For $\forall v_i \in V_1(H)$, $deg(v_i) = 4$ and for $\forall u_i \in V_2(H)$, $deg(u_i) = 2$. Since $deg(v_i) > deg(u_i)$, we consider vertices $V_1(H)$ for the formation of $\gamma_s(H)$ - strong domination value. For $\forall v_i \in V_1(H)$ vertex, graph H will have 4 strong domination values. In this condition, $\lceil \frac{2n}{4} \rceil$ number of values are taken from the set $V_1(H)$. However, in condition $n \equiv_4 2$, a vertex $u_i \in V_2(H)$ with no strong domination value will remain. For this vertex to have a strong domination value, $\exists v_i \in V_1(H)$, $u_i \in N(v_i)$, vertex v_i , $\gamma_s(H)$ -strong domination value must be taken. □

THEOREM 3.6. *If $G = P_n$, then the strong domination number of $S'(G) = H$ is*

$$\gamma_s(H) = \begin{cases} \lceil \frac{2n}{4} \rceil + 1, & n \equiv_4 2 \\ \lceil \frac{2n}{4} \rceil, & \text{otherwise} \end{cases}$$

PROOF. For $v_1, v_n \in V_1(H)$, $deg(v_1) = deg(v_n) = 2$ and $\forall v_i \in V_1(H) - \{v_1, v_n\}$, $deg(v_i) = 4$. For $u_1, u_n \in V_2(H)$, $deg(u_1) = deg(u_n) = 1$ and $\forall u_i \in V_2(H) - \{u_1, u_n\}$, $deg(u_i) = 2$. Since $deg(v_i) > deg(u_i)$, we consider vertices $V_1(H)$ for the formation of $\gamma_s(H)$ - strong domination value. We follow the same procedure of proof like on the Theorem 3.5. □

THEOREM 3.7. *If $G = K_n$, then the strong domination number of $S'(G) = H$ is*

$$\gamma_s(H) = 2$$

PROOF. For $\forall v_i \in V_1(H)$, $deg(v_i) = 2n - 2$, For $\forall u_i \in V_2(H)$, $deg(u_i) = n - 1$. Since $deg(v_i) > deg(u_i)$, we consider vertices $V_1(H)$ for the formation of $\gamma_s(H)$ -strong domination value. For every vertex $v_i \in V_1(H)$, if we take a set of strong domination values, apart from one vertex, all vertices in the sets $V_1(H)$ and $V_2(H)$ will have a strong dominating values. Non-dominated vertex, corresponding vertex

v_i is $u_i \in V_2(H)$ vertex. For $\exists v_i \in V_1(H)$, $u_i \in N(v_i)$, vertex v_i $\gamma_s(H)$ - strong dominating values must be taken. In this condition we get

$$\gamma_s(H) = 2$$

□

THEOREM 3.8. *If $G = K_{1,n}$, then the strong domination number of $S'(G) = H$ is*

$$\gamma_s(H) = 2$$

PROOF. The set of vertices $V_1(H)$ is formed by vertices of graph $K_{1,n}$. From here, the degree of the central vertex c is $deg(c) = 2n$. For remaining vertices, $\forall v_i \in V_1(H) - \{c\}$, degree is $deg(v_i) = 2$. When forming graph H , the degree of c , corresponding central vertex in the set $V_2(H)$ is given by, $deg(c') = n$. For other vertices $\forall u_i \in V_2(H) - \{c'\}$, the degree is given as $deg(u_i) = 1$.

Strong domination set is formed from $V_1(H)$ set. When we take central vertex $c \in V_1(H)$ to strong domination set, all vertices in the set $V_1(H)$ and $V_2(H) - \{c'\}$ will have domination. In this condition, we have to take non-dominant vertices $c' \in V_2(H)$ to the strong domination set. This results to

$$\gamma_s(H) = 2$$

□

THEOREM 3.9. *If $G = W_{1,n}$, then the strong domination number of $S'(G) = H$ is*

$$\gamma_s(H) = 2$$

PROOF. By labelling the central vertex c in the set with v_1, v_2, \dots, v_n , we have $deg(c) = 2n$ and $deg(v_i) = 4 \quad i = \overline{1, n}$. Labelling corresponding central vertex c' in the set $V_2(H)$ from H with u_1, u_2, \dots, u_n , we have $deg(c') = n$, $deg(u_i) = 2 \quad i = \overline{1, n}$. We consider vertex $c \in V_1(H)$ when forming a strong domination set. In this condition, all vertices in set $V_1(H)$ and $V_2(H) - \{c'\}$ will have domination. The only remaining non-domination vertex is $c' \in V_2(H)$. This vertex must also be taken to the strong domination set. Therefore, we have

$$\gamma_s(H) = 2$$

□

COROLLARY 3.1. *If $G = C_n, P_n, K_n, K_{1,n}$ and $W_{1,n}$, then the domination number of $S'(G) = H$ is $\gamma(H) = \gamma_s(H)$.*

4. Conclusion

The integrity, domination and vulnerability of network are important concepts for the network security. We have studied important measures of vulnerability known as integrity, domination and strong domination number and investigate integrity, domination and strong domination number of splitting graphs of path, cycle, complete, star and wheel. The results reported here throw some light in the

direction to find the integrity, domination and strong domination number of larger graph obtained from the given graph.

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EGE UNIVERSITY, FACULTY OF ENGINEERING, DEPARTMENT OF COMPUTER ENGINEERING,
35100 BORNOVA, IZMIR-TURKEY

E-mail address: : `vecdi.aytac@ege.edu.tr` - `sema.bodur@ege.edu.tr`