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HUB-INTEGRITY OF GRAPHS

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Abstract. In this paper the concept of hub-integrity is introduced as a new measure of the stability of a graph G and it is defined as $HI(G) = \min\{|S| + m(G-S)\}$, where S is hub set and $m(G-S)$ is the order of a maximum component of $G-S$. In this paper, the hub-integrity of some graphs is obtained. The relations between hub-integrity and other parameters are determined.

1. Introduction

Let $G = (V, E)$ be a simple graph with p vertices and q edges. The symbols $\Delta(G)$, $\delta(G)$, $\alpha(G)$, $\kappa(G)$, $\lambda(G)$, $\beta(G)$ and $\chi(G)$ denote the maximum degree, the minimum degree, the vertex cover number, the connectivity, the edge-connectivity, the independence number and chromatic number of G, respectively. For graph theoretic terminology, we refer to [8].

In an analysis of the vulnerability of a communication network to disruption, two qualities that come to mind are the number of elements that are not functioning and the size of the largest remaining subnetwork within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two qualities can be made to be simultaneously small.

The integrity of a graph $G = (V, E)$, which was introduced in [3] as a useful measure of the vulnerability of the graph, is defined as follows: $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of the largest component. Barefoot, Entringer and Swart [4] defined the edge-integrity of a graph G with edge set $E(G)$ by $I'(G) = \min\{|S| + m(G-S) : S \subseteq E(G)\}$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. In [3] Barefoot et al gave some basic results on integrity. In [6] Moazzami et al compared the integrity, connectivity, binding

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number, toughness, and tenacity for several classes of graphs. To know more about integrity and Edge-integrity one can see [1, 2, 4, 7].

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$ [14]. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [9]. We need the following to prove main results.

Lemma 1.1. ([14]) *For any graph G, $\gamma(G) \leqslant h(G) + 1$.*

Theorem 1.1. ([7]) *For any graph G, $I(G) \geqslant \chi(G)$.*

Theorem 1.2. ([13]) *For any graph G, $\lceil \frac{p}{1+\Delta(G)} \rceil \leqslant \gamma(G)$, where $\lceil x \rceil$ is the least integer not less than x.*

Theorem 1.3. ([8]) *For any nontrivial connected graph G, $\alpha(G) + \beta(G) = p$.*

Theorem 1.4. ([11]) *If T is a binary tree with n terminal vertices, then T has $n - 1$ internal vertices.*

Theorem 1.5. ([8]) *For any graph G, $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.*

Theorem 1.6. ([10]) *For any graph G, $p-q \leqslant \gamma(G)$. Furthermore, $\gamma(G) = p-q$ if and only if each component of G is a star.*

2. Hub-integrity of graphs

Definition 2.1. The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G - S)\}$, where S is a hub set and $m(G - S)$ is the order of a maximum component of $G - S$.

A HI-set of G is any subset S of $V(G)$ for which $HI(G) = |S| + m(G - S)$. For any disconnected graph G having k components $G_1, G_2, ..., G_k$ of orders $p_1, p_2, ..., p_{k-1}, p_k$, respectively such that $p_1 \leqslant p_2 \leqslant .... \leqslant p_{k-1} \leqslant p_k$. We have $HI(G) = p_1 + p_2 + ... + p_{k-1} + HI(G_k)$. Also, by the definition of hub-integrity we obtain the obvious bound $HI(G) \geqslant I(G)$.

We now proceed to compute $HI(G)$ for some standard graphs.

Proposition 2.1.

(1) For any complete graph K_p, $HI(K_p) = p$.
(2) For any path P_p with $p \geqslant 3$, $HI(P_p) = p - 1$.
(3) For any cycle C_p,

$$HI(C_p) = \begin{cases} p-1, & \text{if } p = 4, 5 \ ; \\ p-2, & \text{if } p \geqslant 6. \end{cases}$$

(4) For the star $K_{1,n}$, $HI(K_{1,n}) = 2$.
(5) For the double star $S_{n,m}$, $HI(S_{n,m}) = 3$.
(6) For the complete bipartite graph $K_{n,m}, HI(K_{n,m}) = \min\{n, m\} + 1$.
(7) For the wheel graph $W_{1,n}$,

$$HI(W_{1,n}) = \begin{cases} n, \text{ if } n \leqslant 4 \ ; \\ \lceil \frac{n}{3} \rceil + 3, \text{ if } n \geqslant 5. \end{cases}$$

(8) For the complete k-bipartite graph $K_{n_1,n_2,.......,n_k}$,

$$HI(K_{n_1,n_2,.......,n_k}) = \sum_{i=1}^{k} n_i + 1 - \max_{i=1}^{k} n_i.$$

Remark 2.1. In general, the inequality $HI(G') \leqslant HI(G)$ is not true for a subgraph G' of G, For example, for the graph G and a subgraph G' of G shown in Figure 1, we have $HI(G) = 6$, while $HI(G') = 7$.

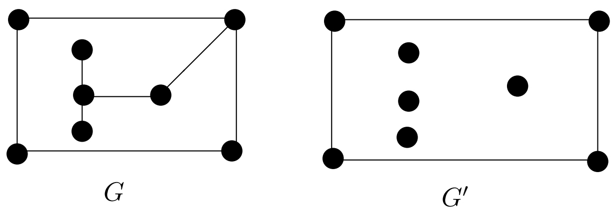


Figure 1

Theorem 2.1. *For any subset D of vertices in a graph G,*

$$HI(G - D) \geqslant HI(G) - |D|.$$

Proof. Let S be a HI-set of G, let $D \subseteq V(G)$ and $S^\star$ be a HI-set of $G - D$ such that $S^{\star\star} = S^\star \cup D$. Then $|S^{\star\star}| = |S^\star| + |D|$ and $G - S^{\star\star} = G - (S^\star \cup D) = (G - D) - S^\star$. Therefore,

$$\begin{aligned} HI(G) &= |S| + m(G - S) \\ &\leqslant |S^{\star\star}| + m(G - S^{\star\star}) \\ &= |S^\star| + |D| + m[(G - D) - S^\star] \\ &= HI(G - D) + |D|. \end{aligned}$$

□

Proposition 2.2. *If G is a non-trivial graph, then for all $v \in V(G)$,*

$$HI(G - v) \geqslant HI(G) - 1.$$

The bound sharp for $G = K_p$.

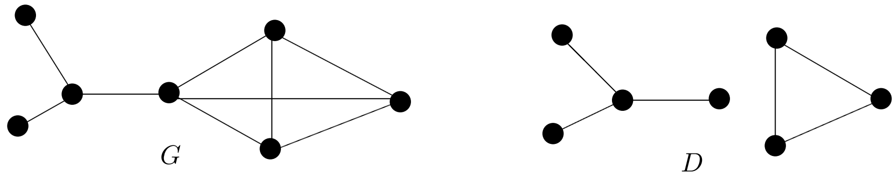
Proposition 2.3. *Let G be a graph. Then for all $e \in E(G)$,*

$$HI(G-e) \geqslant HI(G) - 1$$

.
The bound sharp for $G = K_p$

Lemma 2.1. *For any graph G,*

(1) *If G is non-complete, then every $HI-$set of G is a cut-set of G and hence has cardinality at least $\kappa(G)$.*
(2) *If D is a spanning subgraph of G and $HI(G) = HI(D)$, then not necessary a hub set of G is a hub set of D.*
For example,



$HI(G) = HI(D) = 5$, but a hub set of G is 2, while a hub set of D is 4.

Proposition 2.4. *For any graph G , $1 \leqslant HI(G) \leqslant p$.*
The lower bound attains for K_1 and the upper bound attains for a complete graph K_p, $p \geqslant 2$.

Theorem 2.2. *For any graph G, $HI(G) \geqslant \delta(G) + 1$.*

Proof. Let S be a HI-set of G such that $HI(G) = |S| + m(G-S)$. Then $m(G-S) \geqslant \delta(G-S) + 1 \geqslant \delta(G) - |S| + 1$, So, $HI(G) = |S| + m(G-S) \geqslant |S| + \delta(G) - |S| + 1 = \delta(G) + 1$. □

Theorem 2.3. *For any graph G, $HI(G) \geqslant \lambda(G) + 1$.*

Proof. Proof follows from Theorem 1.5 and Theorem 2.2. □

Corollary 2.1. *For any connected graph G. If $\delta(G) = \alpha(G)$, then*

$$HI(G) = \delta(G) + 1 = \alpha(G) + 1$$

.

Theorem 2.4. *For any tree T, $HI(T) \geqslant \alpha(T) + 1$.*

Proof. Let S' be a minimum covering set of T. Then

$$\begin{aligned} HI(T) &= |S| + m(T-S) \\ &\geqslant |S'| + m(T-S')(Because\ S \geqslant S') \\ &\geqslant |S'| + 1 \\ &= \alpha(T) + 1. \end{aligned}$$

□

Remark 2.2. The hub-integrity and edge-integrity of graph G are not comparable. For this situation consider the graphs in the following cases:

- In the complete graph, we have $HI(K_p) = I'(K_p) = p$.
- In the stare $K_{1,n}$, $HI(K_{1,n}) < I'(K_{1,n})$, $n > 1$.
- In the cycle C_{10}, $HI(C_{10}) > I'(C_{10})$, since $HI(C_{10}) = 8$ and $I'(C_{10}) = 7$.

Theorem 2.5. *For any connected graph G, $HI(G) = \kappa(G) + 1$ if and only if $\kappa(G) = \alpha(G)$.*

Proof. Suppose that $HI(G) = \kappa(G) + 1$. Let S be a HI-set of a graph G such that $HI(G) = |S| + m(G - S)$. If G is complete, then by proposition 2.1, $HI(G) = \kappa(G) + 1$. Thus we may assume that G is non-complete. Since $HI(G) = |S| + m(G - S) = \kappa(G) + 1$, it follows by lemma 2.1, that $|S| \geqslant \kappa(G)$. Thus $\kappa(G) + m(G - S) \leqslant \kappa(G) + 1$. Therefore, $m(G - S) \leqslant 1$. Since $m(G - S) > 0$, it follows that $m(G - S) = 1$. So, we have the following cases:
Case 1: $m(G - S) = 1$, then S is a cover set and we have $|S| = \alpha(G)$.
Case 2: $HI(G) = |S| + m(G - S) = \kappa(G) + 1$ and $m(G - S) = 1$. Then $|S| = \kappa(G)$. Consequently, $\kappa(G) = \alpha(G)$.
Conversely, let S be a hub set of a graph G. Then we have the following cases:
Case 1: $|S| < \kappa(G)$. By lemma 2.1, $G = K_p$. Therefore,

$$HI(K_p) = p. \quad (1)$$

Case 2: $|S| = \kappa(G)$. Since $\kappa(G) = \alpha(G)$, it follows that $m(G - S) = 1$. So,

$$HI(G) = \kappa(G) + 1. \ (2)$$

Case 3: $|S| > \kappa(G)$. Since $\kappa(G) = \alpha(G)$, it follows that $m(G - S) \geqslant 1$ and so

$$HI(G) \geqslant \kappa(G) + 2. \ (3)$$

Since $\kappa(G) \leqslant p - 1$ for every graph G, we have by using (1),(2) and (3)
$HI(G) = \min\{p, \kappa(G) + 1, \kappa(G) + 2\} = \kappa(G) + 1$. □

Theorem 2.6. *Let G be a connected graph of order $p > 1$. Then $HI(G) = 2$ if and only if $\alpha(G) = 1$.*

Proof. Let S be a HI-set of G. Since $HI(G) = |S| + m(G - S) = 2$ and $m(G - S) \geqslant 1$ it follows that $|S| = 1$ and $m(G - S) = 1$. Thus, $|S| = \alpha(G) = 1$.
Converesly, suppose $\alpha(G) = 1$. Then $G \cong K_{1,n}$. Thus, $HI(G) = 2$. □

Theorem 2.7. *Let G be a non-complete graph of order p. If $\beta \leqslant 2$, then $HI(G) = p - 1$.*

Proof. Let $\beta \leqslant 2$. Then $p - \beta(G) \geqslant p - 2$. By Theorem 1.3, $\alpha(G) + \beta(G) - \beta(G) \geqslant p - 2$. Therefore, $\alpha(G) \geqslant p - 2$. Since $|S| \geqslant \alpha(G)$ for any hub set, it follows that $h(G) \geqslant p - 2$. Then $h(G) = p - 2$ or $h(G) = p - 1$ or $h(G) = p$. If $h(G) = p - 1$, then G is complete graph, a contradiction. Since $h(G) \neq p$, we have $h(G) = p - 2$. So $|S| = p - 2$. If we delete the vertices of hub set from a graph G, we get either two components of order 1 or a K_2, So $m(G - S) = 1$ *or* 2. Thus, $HI(G) = p - 2 + 1 = p - 1$. □

THEOREM 2.8. *For any graph G, $\gamma(G) \leqslant HI(G)$.*

PROOF. By the definition of $HI(G)$, $h(G)+1 \leqslant HI(G)$ and by lemma 1.1, $\gamma(G) \leqslant h(G)+1 \leqslant HI(G)$. Therefore $\gamma(G) \leqslant HI(G)$. □

THEOREM 2.9. *For any graph G, $\lceil \frac{p}{1+\Delta(G)} \rceil \leqslant HI(G)$. The bound is sharp for $G = \overline{K}_p$.*

PROOF. Proof follows from Theorem 1.2 and Theorem 2.8. □

PROPOSITION 2.5. *For any graph G, $G \neq \overline{K}_p$, $\gamma(G)+1 \leqslant HI(G)$. The bound is sharp for $G = K_{1,n}$.*

THEOREM 2.10. *For any graph G, $HI(G) = \gamma(G)$ if and only if $G = \overline{K}_p$.*

PROOF. Suppose that $HI(G) = \gamma(G)$. Then $E(G) = \phi$ (since by proposition 2.5, if $E(G) \neq \phi$, then $\gamma(G)+1 \leqslant HI(G)$). Hence $G = \overline{K}_p$.
The converse is obvious. □

THEOREM 2.11. *For any $G(p,q)$ graph, $HI(G) \geqslant p-q$.*

PROOF. Proof follows from Theorem 1.6 and Theorem 2.8 □

THEOREM 2.12. *For any graph G, $HI(G) \geqslant \chi(G)$.*

PROOF. Proof follows from Theorem 1.1. □

THEOREM 2.13. *Let G be a connected graph with $\Delta(G) \leqslant 2$. Then*

$$HI(G) = |E(G)| \ if \ and \ only \ if \ G = P_p.$$

PROOF. Suppose that G is a connected graph with $\Delta(G) \leqslant 2$. Then G is path or cycle. But if G is cycle, we have $HI(G) \leqslant p-1 \neq |E(C_p)|$. Thus, G is path.
The converse is obvious. □

THEOREM 2.14. *For any graph G,*

(1) $p+2 \leqslant HI(G) + HI(\overline{G}) \leqslant 2p$.
(2) $2p \leqslant HI(G).HI(\overline{G}) \leqslant p^2$.

The lower bound attains for complete graph K_p $p \geqslant 2$ and the upper bound attains for a star $K_{1,n}$.

THEOREM 2.15. *Let T be a tree with p vertices and n terminals vertices. Then $HI(G) = p-n+1$.*

PROOF. Let $HI(T) = |S| + m(S-T)$. The set $p-n$ of all internals vertices in T forms a hub set, since the unique path between any two terminals never passes through another terminal. Note that any proper subset of $p-n$ cannot be a hub set. So $|S| = p-n.$, since every internal vertex is a cut-vertex. If we delete of all $p-n$ vertices, we get two component of order 1. So, $HI(T) = |S| + m(T-S) = p-n+1$. □

Definition 2.2. ([12]) The binomial tree B_p is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_p consists of two binomial trees B_{p-1} that are linked together: the root of one is the leftmost child of the root of the other.

Theorem 2.16. *Let $n \geqslant 1$ be a positive integer. Then $HI(B_n) = |V(B_{n-1}|+1$.*

Proof. Let S be a H-set of B_n. Let k the number of internal vertices in B_n. Since the internal vertices in B_n is a minimum hub set of B_n, it follows that $|S| = k$. Since for any binomial tree B_n the number of internal vertices equal to the number of vertices in B_{n-1}, it follows that $|S| = |V(B_{n-1})|$. But the removal S from B_n results a totally disconnected graph. Therefore, $HI(B_n) = |V(B_{n-1}| + 1$. □

Definition 2.3. ([11]) A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices of degree 1 or 3. clearly, P_3 is the smallest binary tree.

Theorem 2.17. *If a tree T is a binary tree of order p. Then $HI(T) = \lceil \frac{p}{2} \rceil$.*

Proof. Let $HI(T) = |S| + m(G - S)$. Since the hub set in any binary tree is internal vertices, by Theorem 1.4, $|S| = n - 1$, where n is the set of its number of terminal vertices of T. If we remove $n-1$ internal vertices from binary tree T we get a totally disconnected graph. So, $m(T-S) = 1$. Therefore, $HI(T) = n-1+1 = n$. Since the number of terminal vertices in any binary tree equal $\lceil \frac{p}{2} \rceil$, it follows that $n = \lceil \frac{p}{2} \rceil$, Therefore $HI(T) = n = \lceil \frac{p}{2} \rceil$. □

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