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$d-$congruences of Almost Distributive Lattices

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ABSTRACT. In this paper two kinds of congruences are proposed in an Almost distributive lattice(ADL), first one is considered in terms of ideals generated by derivations and second one is followed by in terms of images of derivations. An equivalent condition is derived for the corresponding quotient ADL to become a Boolean algebra. An another equivalent condition is also established for the existence of a derivation.

1. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy and Rao [**10**] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Swamy, G.C. Rao and G.N. Rao introduced the concept of Stone ADL and characterized it in terms of its ideals. H.E. Bell, L.C. Kappe [**2**] and K. Kaya [**5**] have studied derivations in rings and prime rings after Posner [**6**] had given the definition of the derivation in ring theory. Szasz have introduced and developed the theory of derivations in lattice structure. In a series of papers [**11**] and [**12**] he established the main properties of derivations of lattices. L. Ferrari [**3**] extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations. G. Birkhoff [**1**], George Gratzer, G. Szasz and many authors have studied about various types of ideals and

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congruences all intimated to some extent the behavior of ideals in a distributive lattice. In [**7**], Rafi, Ravi Kumar and Rao introduced the concept of $d-$ideal in an ADL, where d is derivation on ADL. In that paper they have studied the structure of certain classes of ideals in an ADL with respect to a derivation. The concepts of $d-$ideals and injective ideals are introduced. A necessary and sufficient condition is established for a $d-$ideal to become an injective ideal. They have obtained the relations between the class of all injective ideals and the class of all $d-$prime ideals. Recently, Sambasiva rao [**9**] was introduced the concepts of congruences with the help of derivation in distributive lattice and studied their properties. In this paper we introduced two types of congruences in an ADL. First congruence is defined with the help of ideal generated by derivation and second congruence is given by derivation image. We introduced the term θ_d and derived that θ_d is a congruence relation on an ADL. We defined kernel elements in an ADL with respect to derivation and proved that the set K_d of all kernel elements of an ADL is a filter of an ADL. It is observed that θ_d is the largest congruence relation having congruence class K_d when ever quotient ADL L/θ_d is Boolean algebra. Also we introduced an another term θ^d is given by derivation image and observed that θ^d is a congruence relation on an ADL. In addition to this, we proved that $Ker\ \theta^d = Ker\ d$. Finally, we observed that an equivalent condition is obtained for the existence of a derivation.

2. Preliminaries

Definition 2.1. [**10**] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:
1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for all $x, y, z \in L$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leqslant b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leqslant$ is a partial ordering on L.

Theorem 2.1. [**10**] *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$

(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leqslant a \vee b$ *and* $a \wedge b \leqslant b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leqslant c,\ b \leqslant c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a.$

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.2. [**10**] *Let* $(L, \vee, \wedge, 0)$ *be an ADL with 0. Then the following are equivalent:*
1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$*, for all* $a, b \in L$
3). $a \wedge b = b \wedge a$*, for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a \Rightarrow m = a$.

Theorem 2.3. [**10**] *Let* L *be an ADL and* $m \in L$. *Then the following are equivalent:*
1). m *is maximal with respect to* $\leqslant$
2). $m \vee a = m$*, for all* $a \in L$
3). $m \wedge a = a$*, for all* $a \in L$
4). $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [**1, 4**], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] :=$

$\{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

THEOREM 2.4. [**10**] *For any x, y in L the following are equivalent:*
1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

THEOREM 2.5 ([**8**]). *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

3. $d-$congrurnces of ADLs

In [**9**], M.S. Rao was introduced the concepts of congruences with the help of derivation in distributive lattice and studied their properties. We introduced θ_d to an ADL,analogously. Though many results look similar, the proofs are not similar because of the lack of the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL. Through out this paper L represents an ADL with 0.

First, we recall the following definition and result.

DEFINITION 3.1. [**7**] Let L be an ADL. A self-mapping $d : L \longrightarrow L$ is called a derivation of L if it satisfies the following properties:
(i). $d(x \wedge y) = d(x) \wedge y$
(ii). $d(x \vee y) = d(x) \vee d(y)$, for all $x, y \in L$.

The kernel of a derivation is defined as $Ker\ d = \{x \in L \mid d(x) = 0\}$.
We have the following result.

LEMMA 3.1. [**7**] *Let L be an ADL and d, any derivation of L. Then we have*
(1). $d(0) = 0$
(2). $d^2(x) = d(x)$
(3). $d(x) \leqslant x$, *for all $a \in L$*
(4). *$Ker\ d$ is an ideal of L.*

PROOF. Now $d^2(x) = d(d(x)) = d(d(x \wedge x)) = d(d(x) \wedge x) = d(x \wedge d(x) \wedge x) = d(x \wedge d(x)) = d(x) \wedge d(x) = d(x)$. □

Now we have the following definition of θ_d.

DEFINITION 3.2. Let d be a derivation of an ADL L. For any $a \in L$, define the set $(a)^d = \{x \in L \mid x \wedge a \in Ker\ d\}$.

We prove the following properties.

LEMMA 3.2. *Let d be a derivation of an ADLL. Then for any $a, b, c \in L$, the following conditions hold:*
1. $Ker\ d \subseteq (x)^d$, *for all* $x \in L$
2. *If* $a \in Ker\ d$ *then* $(a)^d = L$
3. *If* $a = 0$ *then* $a \in (a)^d$
4. $(a)^d$ *is an ideal of* L
5. *If* $a \leqslant b$ *then* $(b)^d \subseteq (a)^d$
6. $(a \vee b)^d = (b \vee a)^d$ *and* $(a \wedge b)^d = (b \wedge a)^d$
7. $(a \vee b)^d = (a)^d \cap (b)^d$
8. *If* $(a)^d = b^d$ *then* $(a \wedge c)^d = (b \wedge c)^d$ *and* $(a \vee c)^d = (b \vee c)^d$.

PROOF. 1. Let $t \in Ker\ d$ and $x \in L$. Then $t \wedge x \in Ker\ d$, since $Ker\ d$ is an ideal of L. That implies $t \in (x)^d$. Therefore $Ker\ d \subseteq (x)^d$, for all $x \in L$.
2. Let $a \in Ker\ d$. Since $Ker\ d$ is an ideal of L, we get $x \wedge a \in Ker\ d$, for all $x \in L$. Therefore $x \in (a)^d$ and hence $(a)^d = L$.
3. Clear.
4. Clearly $0 \in (a)^d$. Therefore $(a)^d \neq \emptyset$. Let $x, y \in (a)^d$. Then $x \wedge a, y \wedge a \in Ker\ d$. Since $Ker\ d$ is an ideal of L, we get $(x \vee y) \wedge a \in Ker\ d$ and hence $x \vee y \in (a)^d$. Let $x \in (a)^d$ and $r \in L$. Then $x \wedge a \in Ker\ d$ and hence $x \wedge r \wedge a \in Ker\ d$. Therefore $x \wedge r \in (a)^d$. Thus $(a)^d$ is an ideal of L.
5. Let $a \leqslant b$. we prove that $(b)^d \subseteq (a)^d$. Let $x \in (b)^d$. Then $x \wedge b \in Ker\ d$ and hence $x \wedge b = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) \in Ker\ d$. Since $Ker\ d$ is an ideal of L, we get $x \wedge a \in Ker\ d$. Therefore $x \in (a)^d$. Thus $(b)^d \subseteq (a)^d$.
6. It is obtained easily.
7. Clearly we have $(a \vee b)^d \subseteq (a)^d \cap (b)^d$. Let $x \in (a)^d \cap (b)^d$. Then $x \wedge a,\ x \wedge b \in Ker\ d$ and hence $x \wedge (a \vee b) \in Ker\ d$. That implies $x \in (a \vee b)^d$. Therefore $(a)^d \cap (b)^d \subseteq (a \vee b)^d$. Thus $(a \vee b)^d = (a)^d \cap (b)^d$.
8. Assume that $(a)^d = (b)^d$. Now, $x \in (a \wedge c)^d \Leftrightarrow x \wedge a \wedge c \in Ker\ d \Leftrightarrow x \wedge c \wedge a \in Ker\ d \Leftrightarrow x \wedge c \in (a)^d = (b)^d \Leftrightarrow x \wedge c \wedge b \in Ker\ d \Leftrightarrow x \wedge b \wedge c \in Ker\ d \Leftrightarrow x \in (b \wedge c)^d$. Therefore $(a \wedge c)^d = (b \wedge c)^d$. Now we prove that $(a \vee c)^d = (b \vee c)^d$. Let $x \in (a \vee c)^d$. Then $x \wedge (a \vee c) \in Ker\ d \Rightarrow (x \wedge a) \vee (x \wedge c) \in Ker\ d \Rightarrow x \wedge a, x \wedge c \in Ker\ d \Rightarrow x \in (a)^d = (b)^d \Rightarrow x \wedge b \in Ker\ d$ and hence $(x \wedge b) \vee (x \wedge c) \in Ker\ d$. That implies $x \wedge (b \vee c) \in Ker\ d$. Therefore $x \in (b \vee c)^d$. Hence $(a \vee c)^d \subseteq (b \vee c)^d$. Similarly, we get $(b \vee c)^d \subseteq (a \vee c)^d$. Therefore $(a \vee c)^d = (b \vee c)^d$. □

We have the following definition.

DEFINITION 3.3. Let d be a derivation of L. for any $x, y \in L$, define a relation on L with respect to d, as $(x, y) \in \theta_d$ iff $(x)^d = (y)^d$.

It is observed that θ_d is a congruence relation on L. For any $x \in L$, the congruence class $\theta(x)$ of L with respect to θ, i.e. $\theta(x) = \{t \in L \mid (x, t) \in \theta\}$. Let us denote the set of all congruence classes of L by L_θ.
Now we prove the following result.

THEOREM 3.1. *Let θ be a congruence relation on an ADL L and $L_\theta = \{\theta(x) \mid x \in L\}$. Define binary operations $\vee, \wedge$ on L_θ by $\theta(x)\wedge\theta(y) = \theta(x\wedge y)$ and $\theta(x) \vee \theta(y) = \theta(x \vee y)$, then $(L_\theta; \vee, \wedge)$ is an ADL.*

PROOF. Let θ be a congruence relation on an ADL L. For any $x \in L$, $\theta(x) = \{y \in L \mid (x, y) \in \theta\}$. Write $L_\theta = \{\theta(x) \mid x \in L\}$. Define binary operations $\vee, \wedge$ on L_θ by $\theta(x) \wedge \theta(y) = \theta(x \wedge y)$ and $\theta(x) \vee \theta(y) = \theta(x \vee y)$.

1. $\vee, \wedge$ are well defined: Let $\theta(x) = \theta(x_1), \theta(y) = \theta(y_1) \Rightarrow (x, x_1), (y, y_1) \in \theta \Rightarrow (x \vee y; x_1 \vee y_1) \in \theta \Rightarrow (x \vee y)(\theta) = (x_1 \vee y_1)(\theta) \Rightarrow x(\theta) \vee y(\theta) = x_1(\theta) \vee y_1(\theta)$. Similarly, we can prove $\theta(x) \wedge \theta(y) = \theta(x_1) \wedge \theta(y_1)$. Therefore θ is well defined.

2.RD $\wedge$: Let $\theta(x), \theta(y), \theta(z) \in L_\theta$. Consider $(\theta(x)\vee\theta(y))\wedge\theta(z) = \theta(x\vee y)\wedge\theta(z) = \theta((x\vee y)\wedge z)) = \theta((x\wedge z)\vee(y\wedge z)) = \theta(x\wedge z)\vee\theta(y\wedge z) = (\theta(x)\wedge\theta(z))\vee(\theta(y)\wedge\theta(z))$.

(3). LD $\wedge$: Consider $\theta(x) \wedge (\theta(y) \vee \theta(z)) = \theta(x) \wedge \theta(y \vee z) = \theta(x \wedge (y \vee z)) = \theta((x \wedge y) \vee (x \wedge z)) = \theta(x \wedge y) \vee \theta(x \wedge z) = (\theta(x) \wedge \theta(y)) \vee (\theta(x) \wedge \theta(z))$.

(4). LD $\vee$: Consider $\theta(x) \vee (\theta(y) \wedge \theta(z)) = \theta(x) \vee \theta(y \wedge z) = \theta(x \vee (y \wedge z)) = \theta((x \vee y) \wedge (x \vee z)) = \theta(x \vee y) \wedge \theta(x \vee z) = (\theta(x) \vee \theta(y)) \wedge (\theta(x) \vee \theta(z))$.

(5). $\mathbf{A}_1$: Consider $(\theta(x) \vee \theta(y)) \wedge \theta(y) = \theta(x \vee y) \wedge \theta(y) = \theta((x \vee y) \wedge y) = \theta(y)$.

(6). $\mathbf{A}_2$: $(\theta(x) \vee \theta(y)) \wedge \theta(x) = \theta(x)$.

(7). $\mathbf{A}_3$: $\theta(x) \vee (\theta(x) \wedge \theta(y)) = \theta(x)$.

(8). $\theta(x) \vee 0 = \theta(x) \vee \theta(0) = \theta(x \vee 0) = \theta(x)$.

(9). $\theta(x) \wedge 0 = \theta(x) \wedge \theta(0) = \theta(x \wedge 0) = \theta(0) = 0$.
Therefore $(L_\theta, \vee, \wedge, 0)$ is an ADL.

□

We have the following definition.

DEFINITION 3.4. An element x of an ADL L is said to be kernel if $(x)^d = Ker\ d$. The set of all kernel elements of L is denoted by K_d.

Now we have the following.

LEMMA 3.3. *Let L be an ADL with maximal elements.Then for any derivation d of L, we have the following:*
1. *K_d is a congruence class with respect to θ_d*
2. *K_d is closed under $\wedge$ and $\vee$*
3. *K_d is a filter of L.*

PROOF. 1. Clear.
2. Obvious.

3. Let m be any maximal element of an ADL L. Clearly, m is a kernel element of L. So that $K_d \neq \emptyset$. Let $a \in K_d$ and $x \in L$. Then $(a)^d = Ker\ d$. Clearly, $Ker\ d \subseteq (x \vee a)^d$. Let $t \in (x \vee a)^d$. Then $t \wedge (x \vee a) \in Ker\ d$. That implies $t \wedge x, t \wedge a \in Ker\ d$. So that $t \in (a)^d = Ker\ d$. Therefore $(x \vee a)^d \subseteq Ker\ d$ and hence $(x \vee a)^d = Ker\ d$. Thus K_d is a filter of L. □

In the following, a necessary and sufficient condition is derived for the quotient algebra L/θ_d to become a Boolean algebra.

THEOREM 3.2. *Let d be a derivation of L. Then L/θ_d is a Boolean algebra if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y \in Ker\ d$ and $x \vee y \in K_d$.*

PROOF. We first prove that $Ker\ d$ is the smallest congruence class and K_d is the largest congruence class in L/θ_d. Clearly, $Ker\ d$ is a congruence class of L/θ_d. Since $Ker\ d$ is an ideal, we get that for any $a \in Ker\ d$ and $x \in L$, we have $a \wedge x \in Ker\ d$. Hence $\theta_d(a) \wedge \theta_d(x) = \theta_d(a \wedge x) = \theta_d(a) = Ker\ d$. This is true for all $x \in L$. Therefore $\theta_d(a) = Ker\ d$ is the smallest congruence class of L/θ_d. Again, clearly K_d is a congruence class of L/θ_d. Let $a \in K_d$ and $x \in L$. Since K_d is a filter, we get that $x \vee a \in K_d$. Therefore $(x \vee a)^d = Ker\ d$. We now prove that K_d is the greatest congruence class of L/θ_d. For any $a \in K_d$ and $x \in L$, we get that $\theta_d(x) \vee \theta_d(a) = \theta_d(x \vee a) = \theta_d(a)$. Therefore K_d is the greatest congruence class of L/θ_d. We now prove the main part of the Theorem. Assume that L/θ_d is a Boolean algebra. Let $x \in L$ so that $\theta_d(x) \in L/\theta_d$. Since L/θ_d is a Boolean algebra, there exists $\theta_d(y) \in L/\theta_d$ such that $\theta_d(x \wedge y) = \theta_d(x) \cap \theta_d(y) = Ker\ d$ and $\theta_d(x \vee y) = \theta_d(x) \vee \theta_d(y) = K_d$. Hence $x \wedge y \in Ker\ d$ and $x \vee y \in K_d$. Converse can be proved in a similar way. □

THEOREM 3.3. *Let d be a derivation of L. If $L/\theta_d$is a Boolean algebra, then θ_d is the largest congruence relation having congruence class K_d.*

PROOF. Clearly, θ_d is a congruence with K_d as a congruence class. Let θ be any congruence with K_d as a congruence class. Let $(x, y) \in \theta$. Then for any $a \in L$, we can have $(x, y) \in \theta \Rightarrow (x \vee a, y \vee a) \in \theta \Rightarrow x \vee a \in K_d$ iff $y \vee a \in K_d \Rightarrow (x \vee a)^d = Ker\ d$ iff $(y \vee a)^d = Ker\ d \Rightarrow (x)^d \cap (a)^d = Ker\ d$ iff $(y)^d \cap (a)^d = Ker\ d$. Since L/θ_d is a Boolean algebra, there exists $x', a' \in L$ such that $x \wedge x', a \wedge a' \in Ker\ d$ and $(x \vee x')^d = Ker\ d,\ (a \vee a')^d = Ker\ d$. Hence $x' \in (x)^d$ and $a' \in (a)^d$ which implies that $x' \wedge a' \in (x)^d \cap (a)^d = Ker\ d$. Therefore $a' \in (x')^d$. Similarly, we can get $a' \in (y')^d$ for a suitable $y' \in L$. Then, we get
$a' \in (x')^d$ iff $a' \in (y')^d \Rightarrow (x')^d = (y')^d \Rightarrow (x', y') \in \theta_d \Rightarrow x' \in K_d$ iff $y' \in K_d \Rightarrow (x')^d = Ker\ d$iff $(y')^d = Ker\ d \Rightarrow (x \vee x')^d = (x)^d$ iff $(y \vee y')^d = (y)^d \Rightarrow (x)^d = Ker\ d$ iff $(y)^d = Ker\ d \Rightarrow (x)^d = (y)^d \Rightarrow (x, y) \in \theta_d$. □

Now we give the following definition.

DEFINITION 3.5. Let d be a derivation of an ADL L. Then define a relation θ^d with respect to d on L by $(x, y) \in \theta^d$ iff $d(x) = d(y)$, for all $x, y \in L$.

LEMMA 3.4. *For any derivation d of an ADL L, we have the following:*
1. *θ^d is a congruence relation on L*

2. *Ker* $\theta^d = Ker\ d$.

PROOF. 1. Clearly θ^d is an equivalence relation on L and hence it is easy to prove θ^d is congruence on L.
2. $Ker\ \theta^d = \{x \in L \mid (x, 0) \in \theta^d\} = \{x \in L \mid d(x) = d(0) = 0\} = Ker\ d$. □

LEMMA 3.5. *Let d be a derivation of an ADL L. Then we have the following:*
1. $d(x) = x$, *for all* $x \in d(L)$
2. *If* $(x, y) \in \theta^d$ *and* $x, y \in d(L)$, *then* $x = y$.

PROOF. 1. Let $x \in d(L)$. Then $x = d(a)$, for some $a \in L$. That implies $x = d(a) = d(d(a)) = d(x)$. Therefore $d(x) = x$.
2. Let $x, y \in d(L)$ with $(x, y) \in \theta^d$. Then $d(x) = d(y)$ and $x = d(a), y = d(b)$, for some $a, b \in L$. That implies that $x = d(a) = d(y) = d(b) = y$ and hence $x = y$. □

Now, we conclude this paper with the following theorem.

THEOREM 3.4. *Let I be an ideal of an ADL L. Then there exists a derivation d on L such that $d(L) = I$ if and only if there exists a congruence relation θ on L such that $I \cap [x]_\theta$ is a singleton set for all $x \in L$.*

PROOF. Let d be a derivation of L such that $d(L) = I$. For any $x \in L$, we have $d(x) = d(d(x))$. That implies that $(x, d(x)) \in \theta^d$. Hence $d(x) \in I \cap [x]_{\theta^d}$. Therefore $I \cap [x]_{\theta^d} \neq \emptyset$. We prove that $I \cap [x]_{\theta^d}$ is a singleton set. Suppose that $a, b \in I \cap [x]_{\theta^d}$. Then $a, b \in I = d(L)$ and $a, b \in [x]_{\theta^d}$. By the above lemma we get $a = b$. Therefore $I \cap [x]_{\theta^d}$ is a singleton set. Conversely assume that there exists a congruence θ on L such that $I \cap [x]_{\theta^d}$ is a singleton set for any $x \in L$. Then choose x_0 is the single element of $I \cap [x]_{\theta^d}$. Define a map $d : L \longrightarrow L$ by $d(x) = x_0$, for all $x \in L$. Let $a, b \in L$. Then $d(a \vee b) = x_0 = x_0 \vee x_0 = d(a) \vee d(b)$. Now, $d(a \wedge b) = x_0 \in I \cap [x]_{\theta^d}$. Clearly, we have $(d(a), a) \in \theta^d$ and hence $(d(a) \wedge b,\ a \wedge b) \in \theta^d$. That implies $d(a) \wedge b \in I \cap [a \wedge b]_{\theta^d}$. Therefore $d(a \wedge b) = d(a) \wedge b$. Hence d is a derivation on L. □

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