BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 5(2015), 1-12

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

Multiple Positive Solutions for the System of (n, p)-type Fractional Order Boundary Value Problems

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Abstract. In this paper, we establish sufficient conditions for the existence of at least three positive solutions for the system of (n, p)-type fractional order boundary value problems, by employing an Avery generalization of the Leggett–Williams fixed point theorem. And then, we establish the existence of at least $2k - 1$ positive solutions to the fractional order boundary value problems for an arbitrary positive integer k.

1. Introduction

In recent years, the study of fractional order differential equations has emerged as an important area of mathematics. It has wide range of applications in various fields of science and engineering such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity. There has been much attention paid in developing the theory for existence of positive solutions for fractional order differential equations satisfying initial or boundary conditions. To mention a few references, see Miller and Ross [**13**], Podlubny [**14**], Diethelm and Ford [**8**], Kilbas, Srivasthava and Trujillo [**11**] and the references therein. Much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point fractional order boundary value problems (BVPs). To mention the related papers along these lines, we refer to Bai and Lü [**6**], Kauffman and Mboumi [**10**], Benchohra, Henderson, Ntoyuas and Ouahab [**7**], Su and Zhang [**18**], Ahmed and Nieto [**2**], Goodrich [**9**], Rehman and Khan [**17**], Prasad, Murali and Devi [**15**], Prasad and Krushna [**16**].

Motivated by above papers, in this paper, we are concerned with the existence of multiple positive solutions to the coupled system of (n, p)-type fractional order

2010 *Mathematics Subject Classification.* 26A33, 34B15, 34B18.
Key words and phrases. Boundary value problem, Cone, Fractional derivative, Green's function, Positive solution.

differential equations

$$D_{0^+}^{q_1} x(t) + f_1(t, x(t), y(t)) = 0, \ t \in (0, 1), \tag{1.1}$$

$$D_{0^+}^{q_2} y(t) + f_2(t, x(t), y(t)) = 0, \ t \in (0, 1), \tag{1.2}$$

satisfying two-point boundary conditions

$$x^{(j)}(0) = 0, \ 0 \leqslant j \leqslant n-2, \ x^{(p)}(1) = 0, \tag{1.3}$$

$$y^{(j)}(0) = 0, \ 0 \leqslant j \leqslant n-2, \ y^{(p)}(1) = 0, \tag{1.4}$$

where $n-1 < q_1, q_2 \leqslant n$ and $n \geqslant 3$, $1 \leqslant p \leqslant q_1 - 1, q_2 - 1$ is a fixed integer, $f_i : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $D_{0^+}^{q_i}$, for $i = 1,2$ are the standard Riemann–Liouville fractional order derivatives.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous BVP and estimate the bounds for the Green's function. In Section 3, we establish sufficient conditions for the existence of at least three positive solutions to the fractional order BVP (1.1)-(1.4), by using an Avery generalization of the Leggett–Williams fixed point theorem. We also establish the existence of at least $2k-1$ positive solutions to the fractional order BVP (1.1)-(1.4) for an arbitrary positive integer k. In Section 4, as an application, we demonstrate our results with an example.

2. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous BVP and estimate the bounds for the Green's function, which are needed to establish the main results.

Consider the homogeneous BVP corresponding to (1.1), (1.3)

$$-D_{0^+}^{q_1} x(t) = 0, \ t \in (0, 1), \tag{2.1}$$

satisfying the boundary conditions (1.3).

Lemma 2.1. *If $h(t) \in C[0,1]$, then the fractional order differential equation,*

$$D_{0^+}^{q_1} x(t) + h(t) = 0, \ t \in (0, 1), \tag{2.2}$$

satisfying the boundary conditions (1.3)*, has a unique solution,*

$$x(t) = \int_0^1 G_1(t,s) h(s) ds,$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{q_1-1}(1-s)^{q_1-1-p}}{\Gamma(q_1)}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \frac{t^{q_1-1}(1-s)^{q_1-1-p} - (t-s)^{q_1-1}}{\Gamma(q_1)}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.3}$$

Proof. Assume that $x(t) \in C^{[q_1]+1}[0,1]$ is a solution of the fractional order BVP (2.1), (1.3) and is uniquely expressed as

$$I_{0^+}^{q_1} D_{0^+}^{q_1} x(t) = -I_{0^+}^{q_1} h(t)$$

$$x(t)=\frac{-1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}h(s)ds+c_1t^{q_1-1}+c_2t^{q_1-2}+c_3t^{q_1-3}+\cdots+c_nt^{q_1-n}.$$

From $x^{(j)}(0)=0,\ 0\leqslant j\leqslant n-2$, we obtain $c_n=c_{n-1}=c_{n-2}=\cdots=c_2=0$. Then

$$x(t)=\frac{-1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}h(s)ds+c_1t^{q_1-1},$$

$$x^{(p)}(t)=c_1\prod_{i=1}^{p}(q_1-i)t^{q_1-1-p}-\prod_{i=1}^{p}(q_1-i)\frac{1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1-p}h(s)ds.$$

From $x^{(p)}(1)=0$, we have

$$c_1\prod_{i=1}^{p}(q_1-i)-\prod_{i=1}^{p}(q_1-i)\frac{1}{\Gamma(q_1)}\int_0^1(1-s)^{q_1-1-p}h(s)ds=0.$$

Therefore,

$$c_1=\frac{1}{\Gamma(q_1)}\int_0^1(1-s)^{q_1-1-p}h(s)ds.$$

Thus, the unique solution of (2.1), (1.3) is

$$\begin{aligned}x(t)&=\frac{-1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}h(s)ds+\frac{t^{q_1-1}}{\Gamma(q_1)}\int_0^1(1-s)^{q_1-1-p}h(s)ds\\&=\int_0^1G_1(t,s)h(s)ds,\end{aligned}$$

where $G_1(t,s)$ is given in (2.3). □

LEMMA 2.2. *For $t,s\in[0,1]$, the Green's function $G_1(t,s)$ given in (2.3) is nonnegative.*

PROOF. The Green's function $G_1(t,s)$ is given in (2.3). For $0\leqslant s\leqslant t\leqslant 1$,

$$\begin{aligned}G_1(t,s)=&\frac{1}{\Gamma(q_1)}[t^{q_1-1}(1-s)^{q_1-1-p}-(t-s)^{q_1-1}]\\\geqslant&\frac{1}{\Gamma(q_1)}[t^{q_1-1}(1-s)^{q_1-1-p}-t^{q_1-1}(1-s)^{q_1-1}]\\=&\frac{1}{\Gamma(q_1)}[t^{q_1-1}(1-s)^{q_1-1-p}][1-(1-s)^{p}]\geqslant 0.\end{aligned}$$

Clearly, we observe that $G_1(t,s)\geqslant 0,$ for $0\leqslant t\leqslant s\leqslant 1$. □

LEMMA 2.3. *Let $I=[\frac{1}{4},\frac{3}{4}]$. Then the Green's function $G_1(t,s)$ satisfies the following inequalities*

$$G_1(t,s)\leqslant G_1(1,s),\ \ for\ all\ (t,s)\in[0,1]\times[0,1], \tag{2.4}$$

$$G_1(t,s)\geqslant\frac{1}{4^{q_1-1}}G_1(1,s),\ \ for\ all\ (t,s)\in I\times[0,1]. \tag{2.5}$$

Proof. The Green's function $G_1(t,s)$ is given in (2.3). We prove the inequality in (2.4). We define

$$G_1(1,s)=\frac{1}{\Gamma(q_1)}\Big[(1-s)^{q_1-1-p}-(1-s)^{q_1-1}\Big].$$

For $0\leqslant t\leqslant s\leqslant 1$,

$$\frac{\partial G_1(t,s)}{\partial t}=\frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-1-p}\Big]\geqslant 0.$$

Therefore, $G_1(t,s)$ is increasing in t, which implies $G_1(t,s)\leqslant G_1(1,s)$. Now, for $0\leqslant s\leqslant t\leqslant 1$,

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial t}=&\frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-1-p}-(q_1-1)(t-s)^{q_1-2}\Big]\\ \geqslant&\frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-1-p}-(q_1-1)(t-ts)^{q_1-2}\Big]\\ =&\frac{(q_1-1)t^{q_1-2}}{\Gamma(q_1)}\Big[1-\Big(1-(p-1)s+\frac{\overline{p-1}.\overline{p-2}}{2!}s^2+\cdots\Big)\Big](1-s)^{q_1-1-p}\\ =&\frac{(q_1-1)t^{q_1-2}}{\Gamma(q_1)}\Big[(p-1)s+O(s^2)\Big](1-s)^{q_1-1-p}\geqslant 0.\end{aligned}$$

Therefore, $G_1(t,s)$ is increasing in t, which implies $G_1(t,s)\leqslant G_1(1,s)$. Hence the inequality in (2.4) is proved. Now, we establish the inequality in (2.5). For $0\leqslant t\leqslant s\leqslant 1$ and $t\in I$,

$$\begin{aligned}G_1(t,s)=&\frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-1-p}\Big]\\ \geqslant&\frac{t^{q_1-1}}{\Gamma(q_1)}\Big[(1-s)^{q_1-1-p}-(1-s)^{q_1-1}\Big]\\ =&t^{q_1-1}G_1(1,s)\\ \geqslant&\frac{1}{4^{q_1-1}}G_1(1,s).\end{aligned}$$

For $0\leqslant s\leqslant t\leqslant 1$ and $t\in I$,

$$\begin{aligned}G_1(t,s)=&\frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-1-p}-(t-s)^{q_1-1}\Big]\\ \geqslant&\frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-1-p}-(t-ts)^{q_1-1}\Big]\\ =&\frac{t^{q_1-1}}{\Gamma(q_1)}\Big[(1-s)^{q_1-1-p}-(1-s)^{q_1-1}\Big]\\ =&t^{q_1-1}G_1(1,s)\\ \geqslant&\frac{1}{4^{q_1-1}}G_1(1,s).\end{aligned}$$

Hence the inequality in (2.5) is proved. $\square$

In a similar manner, we construct the Green's function $G_2(t,s)$ for the homogeneous fractional order BVP corresponding to the fractional order BVP (1.2), (1.4).

REMARK 2.1. $G_1(t,s) \geqslant \xi G_1(1,s)$ and $G_2(t,s) \geqslant \xi G_2(1,s)$, for all $(t,s) \in [0,1]\times[0,1]$, where $\xi = \min\left\{\frac{1}{4^{q_1-1}}, \frac{1}{4^{q_2-1}}\right\}$.

3. Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the fractional order BVP (1.1)-(1.4), by using an Avery generalization of the Leggett–Williams fixed point theorem. And then, we establish the existence of at least $2k-1$ positive solutions to the fractional order BVP (1.1)-(1.4) for an arbitrary positive integer k.

Let P be a cone in the real Banach space B. A map $\alpha : P \to [0,\infty)$ is said to be nonnegative continuous concave functional on P if α is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \geqslant \lambda\alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$.

Let P be a cone in the real Banach space B. A map $\beta : P \to [0,\infty)$ is said to be nonnegative continuous convex functional on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \leqslant \lambda\beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', we define the following convex sets.

$$\begin{aligned}
P(\gamma, c') &= \{y \in P : \gamma(y) < c'\},\\
P(\gamma, \alpha, a', c') &= \{y \in P : a' \leqslant \alpha(y); \gamma(y) \leqslant c'\},\\
Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \leqslant d'; \gamma(y) \leqslant c'\},\\
P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \leqslant \alpha(y); \theta(y) \leqslant b'; \gamma(y) \leqslant c'\},\\
Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P : h' \leqslant \psi(y); \beta(y) \leqslant d'; \gamma(y) \leqslant c'\}.
\end{aligned}$$

In obtaining multiple positive solutions of the BVP (1.1)-(1.4), the following so called Five Functionals Fixed Point Theorem will be fundamental.

THEOREM 3.1. [**4**] *Let P be a cone in the real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β, θ are non-negative continuous convex functionals on P, such that for some positive numbers c' and e', $\alpha(y) \leqslant \beta(y)$ and $\| y \| \leqslant e'\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants*

h', d', a' and $b' \geqslant 0$ with $0 < d' < a'$ such that each of the following is satisfied.

$(B1)$ $\{y \in P(\gamma, \theta, \alpha, a', b', c') : \alpha(y) > a'\} \neq \emptyset$ *and* $\alpha(Ty) > a'$ *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,

$(B2)$ $\{y \in Q(\gamma, \beta, \psi, h', d', c') : \beta(y) < d'\} \neq \emptyset$ *and* $\beta(Ty) < d'$ *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,

$(B3)$ $\alpha(Ty) > a'$ *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,

$(B4)$ $\beta(Ty) < d'$ *provided* $y \in Q(\gamma, \beta, d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d'$, $a' < \alpha(y_2)$ and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

We consider the Banach space $B = E \times E$, where $E = \{x : x \in C[0,1]\}$ equipped with the norm $\|(x,y)\| = \|x\|_0 + \|y\|_0$, for $(x,y) \in B$ and we denote the norm,

$$\|x\|_0 = \max_{0 \leqslant t \leqslant 1} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{(x,y) \in B \mid x(t) \geqslant 0,\ y(t) \geqslant 0,\ t \in [0,1] \text{ and } \min_{t \in I}[x(t) + y(t)] \geqslant \xi \|(x,y)\|\Big\}.$$

Let $I_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$ and define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\alpha(x,y) = \min_{t \in I}\{|x| + |y|\}, \psi(x,y) = \min_{t \in I_1}\{|x| + |y|\},$$
$$\gamma(x,y) = \max_{t \in [0,1]}\{|x| + |y|\}, \beta(x,y) = \max_{t \in I_1}\{|x| + |y|\}, \theta(x,y) = \max_{t \in I}\{|x| + |y|\}.$$

We observe that for any $(x,y) \in P$, we have

$$\alpha(x,y) = \min_{t \in I}\{|x| + |y|\} \leqslant \max_{t \in I_1}\{|x| + |y|\} = \beta(x,y) \tag{3.1}$$

and

$$\|(x,y)\| \leqslant \frac{1}{\xi} \min_{t \in I}\{|x| + |y|\} \leqslant \frac{1}{\xi} \max_{t \in [0,1]}\{|x| + |y|\} = \frac{1}{\xi}\gamma(x,y). \tag{3.2}$$

Let

$$L = \min\Big\{\int_0^1 G_1(1,s)ds, \int_0^1 G_2(1,s)ds\Big\}$$

and

$$M = \max\Big\{\int_0^1 G_1(1,s)ds, \int_0^1 G_2(1,s)ds\Big\}.$$

We denote the operators $T_1 : P \to E$, $T_2 : P \to E$ and defined by

$$T_1(x,y)(t) = \int_0^1 G_1(t,s) f_1(s, x(s), y(s)) ds,$$
$$T_2(x,y)(t) = \int_0^1 G_2(t,s) f_2(s, x(s), y(s)) ds.$$

THEOREM 3.2. *Suppose there exist* $0 < a' < b' < \frac{b'}{\xi} < c'$ *such that* f_i, *for* $i = 1, 2$ *satisfies the following conditions:*

$$(A1)\ f_i(t,x,y) < \frac{a'}{2M},\ t \in [0,1] \text{ and } |x| + |y| \in [\xi a', a'],$$
$$(A2)\ f_i(t,x,y) > \frac{b'}{2\xi L},\ t \in I \text{ and } |x| + |y| \in \Big[b', \frac{b'}{\xi}\Big],$$
$$(A3)\ f_i(t,x,y) < \frac{c'}{2M},\ t \in [0,1] \text{ and } |x| + |y| \in [0, c'].$$

Then the fractional order BVP (1.1)-(1.4) *has at least three positive solutions,* (x_1, x_2), (y_1, y_2) *and* (z_1, z_2) *such that* $\beta(x_1, x_2) < a', b' < \alpha(y_1, y_2)$ *and* $a' < \beta(z_1, z_2)$ *with* $\alpha(z_1, z_2) < b'$.

PROOF. Define the completely continuous operator $T : P \to B$ by

$$T(x,y)(t) = \Big(T_1(x,y)(t), T_2(x,y)(t)\Big).$$

It is obvious that a fixed point of T is the solution of the fractional order BVP (1.1)-(1.4). We seek three fixed points of T. First, we show that $T : P \to P$. Let $(x, y) \in P$. By Lemma 2.2 and the nonnegativity of f_i, for $i = 1, 2$, we obtain that $T_1(x, y)(t) \geqslant 0, T_2(x, y)(t) \geqslant 0$, for $t \in [0, 1]$. Also, for $(x, y) \in P$,

$$\|T_1(x,y)\|_0 \leqslant \int_0^1 G_1(1,s) f_1(s, x(s), y(s)) ds,$$
$$\|T_2(x,y)\|_0 \leqslant \int_0^1 G_2(1,s) f_2(s, x(s), y(s)) ds,$$

and

$$\begin{aligned}\min_{t\in I} T_1(x,y)(t) =& \min_{t\in I} \int_0^1 G_1(t,s) f_1(s, x(s), y(s)) ds\\ \geqslant& \xi \int_0^1 G_1(1,s) f_1(s, x(s), y(s)) ds\\ \geqslant& \xi \|T_1(x,y)\|_0.\end{aligned}$$

Similarly, $\min_{t\in I} T_2(x, y)(t) \geqslant \xi \|T_2(x, y)\|_0$. Therefore,

$$\begin{aligned}\min_{t\in I} \Big\{T_1(x,y)(t) + T_2(x,y)(t)\Big\} &\geqslant \xi \|T_1(x,y)\|_0 + \xi \|T_2(x,y)\|_0\\ &= \xi(\|T_1(x,y)\|_0 + \|T_2(x,y)\|_0)\\ &= \xi \|(T_1(x,y), T_2(x,y))\|\\ &= \xi \|T(x,y)\|.\end{aligned}$$

Hence, $T(x, y) \in P$ and so $T : P \to P$. Moreover, T is a completely continuous operator. From (3.1) and (3.2), for each $(x, y) \in P$, we have $\alpha(x, y) \leqslant \beta(x, y)$ and $\|(x, y)\| \leqslant \frac{1}{\xi}\gamma(x, y)$. We show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $(x, y) \in \overline{P(\gamma, c')}$.

Then $0 \leqslant |x| + |y| \leqslant c'$. We may use the condition $(A3)$ to obtain

$$\begin{aligned}
&\gamma(T(x,y)(t))\\
&\quad= \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big]\\
&\quad< \frac{c'}{2M}\int_0^1 G_1(1,s)ds + \frac{c'}{2M}\int_0^1 G_2(1,s)ds\\
&\quad\leqslant c'.
\end{aligned}$$

Therefore $T : \overline{P(\gamma,c')} \to \overline{P(\gamma,c')}$. Now we verify the conditions $(B1)$ and $(B2)$ of Theorem 3.1 are satisfied. It is obvious that

$$\Big\{(x,y) \in P\Big(\gamma,\theta,\alpha,b',\frac{b'}{\xi},c'\Big) : \alpha(x,y) > b'\Big\} \neq \emptyset$$

and

$$\Big\{(x,y) \in Q\Big(\gamma,\beta,\psi,\xi a',a',c'\Big) : \beta(x,y) < a'\Big\} \neq \emptyset.$$

Next, let $(x,y) \in P(\gamma,\theta,\alpha,b',\frac{b'}{\xi},c')$ or $(x,y) \in Q(\gamma,\beta,\psi,\xi a',a',c')$. Then, $b' \leqslant |x(t)| + |y(t)| \leqslant \frac{b'}{\xi}$ and $\xi a' \leqslant |x(t)| + |y(t)| \leqslant a'$.
Now, we may apply the condition $(A2)$ to get

$$\begin{aligned}
&\alpha(T(x,y)(t))\\
&\quad= \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big]\\
&\quad\geqslant \xi\Big[\int_0^1 G_1(1,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(1,s)f_2(s,x(s),y(s))ds\Big]\\
&\quad> \frac{b'}{2L}\int_0^1 G_1(1,s)ds + \frac{b'}{2L}\int_0^1 G_2(1,s)ds\\
&\quad\geqslant b'.
\end{aligned}$$

Clearly, by the condition $(A1)$, we have

$$\begin{aligned}
&\beta(T(x,y)(t))\\
&\quad= \max_{t\in I_1}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big]\\
&\quad\leqslant \Big[\int_0^1 G_1(1,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(1,s)f_2(s,x(s),y(s))ds\Big]\\
&\quad< \frac{a'}{2M}\int_0^1 G_1(1,s)ds + \frac{a'}{2M}\int_0^1 G_2(1,s)ds\\
&\quad\leqslant a'.
\end{aligned}$$

To see that $(B3)$ is satisfied, let $(x,y) \in P(\gamma,\alpha,b',c')$ with $\theta(T(x,y)(t)) > \frac{b'}{\xi}$. Then, we have

$$\begin{aligned}
&\alpha(T(x,y)(t)) \\
&\quad = \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \geqslant \xi\Big[\int_0^1 G_1(1,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(1,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \geqslant \xi \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \geqslant \xi \max_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad = \xi\theta(T(x,y)(t)) \\
&\quad > b'.
\end{aligned}$$

Finally, we show that $(B4)$ holds. Let $(x,y) \in Q(\gamma,\beta,a',c')$ with $\psi(T(x,y)(t)) < \xi a'$. Then, we have

$$\begin{aligned}
&\beta(T(x,y)(t)) \\
&\quad = \max_{t\in I_1}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \leqslant \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \leqslant \Big[\int_0^1 G_1(1,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(b,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad = \frac{1}{\xi}\Big[\xi\int_0^1 G_1(1,s)f_1(s,x(s),y(s))ds + \xi\int_0^1 G_2(1,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \leqslant \frac{1}{\xi}\min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad \leqslant \frac{1}{\xi}\min_{t\in I_1}\Big[\int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds\Big] \\
&\quad = \frac{1}{\xi}\psi(T(x,y)(t)) \\
&\quad < a'.
\end{aligned}$$

We have proved that all the conditions of Theorem 3.1 are satisfied. Therefore, the fractional order BVP (1.1)-(1.4) has at least three positive solutions, (x_1,x_2), (y_1,y_2) and (z_1,z_2) such that $\beta(x_1,x_2) < a'$, $b' < \alpha(y_1,y_2)$ and $a' < \beta(z_1,z_2)$ with $\alpha(z_1,z_2) < b'$. This completes the proof of the theorem. $\square$

Now, we establish the existence of at least $2k-1$ positive solutions for the fractional order BVP (1.1)-(1.4), by using induction on k.

THEOREM 3.3. *Let k be an arbitrary positive integer. Assume that there exist numbers $a_r(r = 1, 2, 3, \cdots, k)$ and $b_s(s = 1, 2, 3, \cdots, k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{\xi} < a_2 < b_2 < \frac{b_2}{\xi} < \cdots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{\xi} < a_k$ such that f_i, for $i = 1, 2$ satisfies the following conditions:*

$(A4)$ $f_i(t, x, y) < \dfrac{a_r}{2M}$, $t \in [0, 1]$ *and* $|x| + |y| \in [\xi a_r, a_r]$, $r = 1, 2, 3, \cdots, k$,

$(A5)$ $f_i(t, x, y) > \dfrac{b_s}{2\xi L}$, $t \in I$ *and* $|x| + |y| \in \Big[b_s, \dfrac{b_s}{\xi}\Big]$, $s = 1, 2, 3, \cdots, k-1$.

Then the fractional order BVP (1.1)-(1.4) *has at least* $2k - 1$ *positive solutions in* $\overline{P}_{a_k}$.

PROOF. We use induction on k. First, for $k = 1$, we know from the condition $(A4)$ that $T : \overline{P}_{a_1} \to P_{a_1}$, then it follows from the Schauder fixed point theorem that the fractional order BVP (1.1)-(1.4) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = l$. In order to prove that this conclusion holds for $k = l + 1$, we suppose that there exist numbers $a_r(r = 1, 2, 3, \cdots, l + 1)$ and $b_s(s = 1, 2, 3, \cdots, l)$ with $0 < a_1 < b_1 < \frac{b_1}{\xi} < a_2 < b_2 < \frac{b_2}{\xi} < \cdots < a_l < b_l < \frac{b_l}{\xi} < a_{l+1}$ such that f_i, for $i = 1, 2$ satisfies the following conditions:

$$\begin{cases} f_i(t, x, y) < \dfrac{a_r}{2M}, \ t \in [0, 1] \text{ and } |x| + |y| \in [\xi a_r, a_r], \\ \hfill r = 1, 2, 3, \cdots, l + 1, \end{cases} \tag{3.3}$$

$$\begin{cases} f_i(t, x, y) > \dfrac{b_s}{2\xi L}, \ t \in I \text{ and } |x| + |y| \in \Big[b_s, \dfrac{b_s}{\xi}\Big], \\ \hfill s = 1, 2, 3, \cdots, l. \end{cases} \tag{3.4}$$

By assumption, the fractional order BVP (1.1)-(1.4) has at least $2l - 1$ positive solutions (x_i, x_i^*), $i = 1, 2, 3, \cdots, 2l - 1$ in $\overline{P}_{a_l}$. At the same time, it follows from Theorem 3.2, (3.3) and (3.4) that the fractional order BVP (1.1)-(1.4) has at least three positive solutions (x_1, x_2), (y_1, y_2) and (z_1, z_2) in $\overline{P}_{a_{l+1}}$ such that $\beta(x_1, x_2) < a_l$, $b_l < \alpha(y_1, y_2)$ and $a_l < \beta(z_1, z_2)$ with $\alpha(z_1, z_2) < b_l$. Obviously, (y_1, y_2) and (z_1, z_2) are distinct from (x_i, x_i^*), $i = 1, 2, 3, \cdots, 2l - 1$ in $\overline{P}_{a_l}$. Therefore, the fractional order BVP (1.1)-(1.4) has at least $2l + 1$ positive solutions in $\overline{P}_{a_{l+1}}$ which shows that this conclusion also holds for $k = l + 1$. This completes the proof of theorem. $\square$

4. Example

In this section, as an application, we demonstrate our results with an example.

Consider the system of fractional order two-point boundary value problem,

$$D_{0^+}^{3.5} x(t) + f_1(t, x, y) = 0, \ t \in (0, 1), \tag{4.1}$$

$$D_{0^+}^{3.7} y(t) + f_2(t, x, y) = 0, \ t \in (0, 1), \tag{4.2}$$

$$x(0) = 0, x'(0) = 0, x''(0) = 0 \text{ and } x''(1) = 0, \tag{4.3}$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0 \text{ and } y''(1) = 0, \tag{4.4}$$

where

$$f_1(t,x,y) = \begin{cases} \frac{e^{x+y}}{165} + \frac{19(x+y)^4}{5}, \ 0 \leqslant x+y \leqslant 5, \\ \frac{e^{x+y}}{165} + \frac{(x+y)}{2} + \frac{4745}{2}, \ x+y \geqslant 5, \end{cases}$$

$$f_2(t,x,y) = \begin{cases} \frac{e^{x+y}}{143} + \frac{18(x+y)^4}{5} + \frac{t}{50}, \ 0 \leqslant x+y \leqslant 5, \\ \frac{e^{x+y}}{143} + \frac{t}{50} + \frac{(x+y)}{2} + \frac{4495}{2}, \ x+y \geqslant 5. \end{cases}$$

Then, the Green's functions $G_1(t,s)$ of (4.1),(4.3) and $G_2(t,s)$ of (4.2),(4.4) are given by

$$G_1(t,s) = \begin{cases} \frac{t^{2.5}(1-s)^{0.5}}{\Gamma(3.5)}, & t \leqslant s, \\ \frac{t^{2.5}(1-s)^{0.5}-(t-s)^{2.5}}{\Gamma(3.5)}, & s \leqslant t, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{t^{2.7}(1-s)^{0.7}}{\Gamma(3.7)}, & t \leqslant s, \\ \frac{t^{2.7}(1-s)^{0.7}-(t-s)^{2.7}}{\Gamma(3.7)}, & s \leqslant t. \end{cases}$$

Clearly, the Green's functions $G_1(t,s)$, $G_2(t,s)$ are positive and f_i, for $i = 1,2$ are continuous and increasing on $[0,\infty)$. By direct calculations, one can determine $\xi = 0.02368, L = 0.053234$ and $M = 0.081844$. If we choose $a' = 1.5, b' = 5$ and $c' = 1500$, then $0 < a' < b' < \frac{b'}{\xi} \leqslant c'$ and f_i, for $i = 1,2$ satisfies

$$(i) \ f_i(t,x,y) < 9.16 = \frac{a'}{2M}, \ t \in [0,1], \ x+y \in [0.03, 1.5]$$

$$(ii) \ f_i(t,x,y) > 1983.21 = \frac{b'}{2\xi L}, \ t \in \left[\frac{1}{4}, \frac{3}{4}\right], \ x+y \in [5, 211.14]$$

$$(iii) \ f_i(t,x,y) < 9163.77 = \frac{c'}{2M}, \ t \in [0,1], \ x+y \in [0,1500].$$

Then, all the conditions of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, the fractional order BVP (4.1)-(4.4) has at least three positive solutions.

Acknowledgements: The authors thank the referees for their valuable comments and suggestions.

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Received by editors 18.12.2013. Revised version 07.12.2014; Available online 16.03.2015.

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