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Convergence to common fixed points for generalized asymptotically quasi-nonexpansive mappings

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ABSTRACT. The aim of this article is to establish some strong convergence theorems of three-step iteration process with errors for approximating common fixed point for generalized asymptotically quasi-nonexpansive mappings and also establish a weak convergence theorem by using Opial's [**11**] condition for said iteration scheme and mappings in the framework of Banach spaces. The results presented in this paper extend and improve the corresponding results of [**2, 8, 9, 13, 14, 15, 18**] and many others.

1. Introduction

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [**3**] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [**12**] gave necessary and sufficient conditions for Mann iterative sequence [**10**] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [**2**] extended the results of Petryshyn and Williamson [**12**] and gave necessary and sufficient conditions for Ishikawa iterative sequence [**5**] to converge to fixed points for quasi-nonexpansive mappings.

Liu [**9**] extended results of [**2, 12**] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings. In 2002, Xu and Noor [**18**] introduced and studied a three-step iteration scheme to approximate fixed points of asymptotically nonexpansive mappings.

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Key words and phrases. Generalized asymptotically quasi-nonexpansive mapping, three-step iteration process with errors, common fixed point, strong convergence, weak convergence, Banach space.

In 2006, Quan [**13**] studied some necessary and sufficient conditions for three-step Ishikawa iterative sequences with error terms for uniformly quasi-Lipschitzian mappings to converge to fixed points. The results presented in [**13**] extend and improve the corresponding results of Liu [**8, 9**], Xu and Noor [**18**] and many others.

In 2007, Saluja [**15**] gave the necessary and sufficient condition for the convergence of the modified Ishikawa iterative sequence with errors in the sense of Liu [**7**] involving two asymptotically quasi-nonexpansive mappings in a real Banach space.

Very recently, Imnang and Suantai [**4**] have studied multi-step iteration process for a finite family of generalized asymptotically quasi-nonexpansive mappings and gave a necessary and sufficient condition for the said scheme and mappings to converge to the common fixed points and also they established some strong convergence theorems in the frame work of uniformly convex Banach spaces.

The aim of this paper is to give necessary and sufficient condition of three-step iterative sequence with errors to converge to common fixed points for generalized asymptotically quasi-nonexpansive mappings in the setting of Banach spaces. Also we establish some strong convergence theorems and a weak convergence theorem for said iteration scheme and mappings. The results obtained in this paper extend and improve the corresponding results of [**2, 8, 9, 13, 14, 15, 18**] and many others.

2. Preliminaries

Definition 2.1. Let E be a real Banach space, C be a nonempty convex subset of E and $F(T)$ denotes the set of fixed points of T. Let $T\colon C \to C$ be a mapping:

(1) T is said to be asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leqslant (1+r_n)\|x-y\|, \tag{2.1}$$

for all $x, y \in C$ and $n \geqslant 1$.

(2) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0,\infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leqslant (1+r_n)\|x-p\|, \tag{2.2}$$

for all $x \in C$, $p \in F(T)$ and $n \geqslant 1$.

(3) T is said to be generalized asymptotically quasi-nonexpansive [**4**] if there exist sequences $\{r_n\}$, $\{s_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$\|T^n x - p\| \leqslant (1+r_n)\|x-p\| + s_n, \tag{2.3}$$

for all $x \in C$, $p \in F(T)$ and $n \geqslant 1$.

If $s_n = 0$ for all $n \geqslant 1$, then T is known as an asymptotically quasi-nonexpansive mapping.

(4) T is said to be asymptotically nonexpansive mapping in the the intermediate sense [**1**] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \Big(\|T^n x - T^n y\| - \|x-y\|\Big) \leqslant 0. \tag{2.4}$$

(5) T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leqslant L\|x - y\|, \tag{2.5}$$

for all $x, y \in C$ and $n \geqslant 1$.

REMARK 2.1. Let T be asymptotically nonexpansive mapping in the intermediate sense. Put $G_n = \sup_{x,y\in C}\Big(\|T^n x - T^n y\| - \|x - y\|\Big) \vee 0, \forall n \geqslant 1$.

If $F(T) \neq \emptyset$, we obtain that $\|T^n x - p\| \leqslant \|x - p\| + G_n$ for all $x \in C$ and all $p \in F(T)$. Since $\lim_{n\to\infty} G_n = 0$, therefore T is generalized asymptotically quasi-nonexpansive mapping.

DEFINITION 2.2. Let E be a normed linear space, C be a nonempty convex subset of E, and $T\colon C \to C$ a given mapping. Then for arbitrary $x_1 \in C$, the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined by

$$\begin{aligned} z_n &= (1 - \gamma_n - \nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n, \\ y_n &= (1 - \beta_n - \mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n T_1^n y_n + \lambda_n w_n,\ n \geqslant 1, \end{aligned} \tag{2.6}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are appropriate sequences in $[0, 1]$, is called the three-step iterative sequence with error terms of T.

We note that the usual modified Ishikawa and Mann iterations are special cases of the above three-step iteration scheme. If $\gamma_n = \nu_n = 0$ and $T_1 = T_2$, then (2.6) reduces to the usual modified Ishikawa iterative scheme with errors

$$\begin{aligned} y_n &= (1 - \beta_n - \mu_n)x_n + \beta_n T^n x_n + \mu_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n,\ n \geqslant 1, \end{aligned} \tag{2.7}$$

where $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ are appropriate sequences in $[0, 1]$.

If $\beta_n = \mu_n = 0$, then (2.7) reduces to the usual modified Mann iterative scheme with errors

$$x_1 \in C,$$

$$x_{n+1} = (1 - \alpha_n - \lambda_n)x_n + \alpha_n T^n x_n + \lambda_n w_n,\ n \geqslant 1, \tag{2.8}$$

where $\{w_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0, 1]$.

We say that a Banach space E satisfies the *Opial's condition* [**11**] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [**11**].

DEFINITION 2.3. Let K be a nonempty closed convex subset of a Banach space E. Then $I-T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in K, condition $x_n \to x$ weakly and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $(I-T)x = 0$.

In the sequel, we shall need the following lemma.

LEMMA 2.1. *(See [**17**]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1+\delta_n)a_n + b_n, \qquad n \geqslant 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

In this section, we prove weak and strong convergence theorems of three-step iteration scheme with errors for generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

THEOREM 3.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.6) with the restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\{x_n\}$ converges to a common fixed point of the mappings T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.*

PROOF. The necessity is obvious. Thus we only prove the sufficiency. Let $p \in F$. Using (2.3) and (2.6), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n-\nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n - p\| \\
&\leqslant (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n \|T_3^n x_n - p\| \\
&\quad + \nu_n \|u_n - p\| \\
&\leqslant (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n[(1+r_n)\|x_n - p\| + s_n] \\
&\quad + \nu_n \|u_n - p\| \\
&\leqslant (1+r_n)\|x_n - p\| + \gamma_n s_n + \nu_n \|u_n - p\|, \qquad (3.1)
\end{aligned}$$

again, using (2.6) and (3.1), we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n-\mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n - p\| \\
&\leqslant (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n\|T_2^n z_n - p\| \\
&\quad +\mu_n\|v_n - p\| \\
&\leqslant (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n[(1+r_n)\|z_n - p\| + s_n] \\
&\quad +\mu_n\|v_n - p\| \\
&\leqslant (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n(1+r_n)\|z_n - p\| + \beta_n s_n \\
&\quad +\mu_n\|v_n - p\| \\
&\leqslant (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n(1+r_n)\Big[(1+r_n)\|x_n - p\| + \gamma_n s_n \\
&\quad +\nu_n\|u_n - p\|\Big] + \beta_n s_n + \mu_n\|v_n - p\| \\
&\leqslant (1+r_n)^2\|x_n - p\| + \beta_n s_n(1+r_n)(1+\gamma_n) + \beta_n\nu_n(1+r_n)\|u_n - p\| \\
&\quad +\mu_n\|v_n - p\| \\
&\leqslant (1+r_n)^2\|x_n - p\| + 2\beta_n s_n(1+r_n) + (1+r_n)\nu_n\|u_n - p\| \\
&\quad +\mu_n\|v_n - p\|, \qquad (3.2)
\end{aligned}
$$

by using (2.6) and (3.2), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n-\lambda_n)x_n + \alpha_n T_1^n y_n + \lambda_n w_n - p\| \\
&\leqslant (1-\alpha_n-\lambda_n)\|x_n - p\| + \alpha_n\|T_1^n y_n - p\| \\
&\quad +\lambda_n\|w_n - p\| \\
&\leqslant (1-\alpha_n-\lambda_n)\|x_n - p\| + \alpha_n[(1+r_n)\|y_n - p\| + s_n] \\
&\quad +\lambda_n\|w_n - p\| \\
&\leqslant (1-\alpha_n-\lambda_n)\|x_n - p\| + \alpha_n(1+r_n)\Big[(1+r_n)^2\|x_n - p\| \\
&\quad +2\beta_n s_n(1+r_n) + (1+r_n)\nu_n\|u_n - p\| + \mu_n\|v_n - p\|\Big] \\
&\quad +\alpha_n s_n + \lambda_n\|w_n - p\| \\
&\leqslant (1+r_n)^3\|x_n - p\| + (1+r_n)^2\alpha_n s_n(1+2\beta_n) + \alpha_n\nu_n(1+r_n)^2. \\
&\quad \|u_n - p\| + \alpha_n\mu_n(1+r_n)\|v_n - p\| + \lambda_n\|w_n - p\| \\
&\leqslant (1+r_n)^3\|x_n - p\| + 3\alpha_n s_n(1+r_n)^2 + \alpha_n(1+r_n)^2\|u_n - p\| \\
&\quad +\alpha_n(1+r_n)\|v_n - p\| + \lambda_n\|w_n - p\| \\
&\leqslant (1+A_n)\|x_n - p\| + H_n, \qquad (3.3)
\end{aligned}
$$

where

$$A_n = r_n^3 + 3r_n^2 + 3r_n$$

and

$$H_n = 3\alpha_n s_n(1+r_n)^2 + \alpha_n(1+r_n)^2\|u_n - p\| + \alpha_n(1+r_n)\|v_n - p\| + \lambda_n\|w_n - p\|.$$

Since by hypothesis $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C, it follows that $\sum_{n=1}^{\infty} A_n < \infty$

and $\sum_{n=1}^{\infty} H_n < \infty$. From (3.3) and Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Also from (3.3), we obtain

$$d(x_{n+1}, F) \leqslant (1 + A_n)d(x_n, F) + H_n, \tag{3.4}$$

for all $n \geqslant 1$. From Lemma 2.1 and (3.4), we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we have that $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $1 + x \leqslant e^x$ for $x \geqslant 0$, therefore, for any $m, n \geqslant 1$ and for given $p \in F$, from (3.3), we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leqslant (1 + A_{n+m-1})\|x_{n+m-1} - p\| + H_{n+m-1} \\
&\leqslant e^{A_{n+m-1}}\|x_{n+m-1} - p\| + H_{n+m-1} \\
&\leqslant e^{A_{n+m-1}} e^{A_{n+m-2}}\|x_{n+m-2} - p\| + e^{A_{n+m-1}} H_{n+m-2} \\
&\quad + H_{n+m-1} \\
&\leqslant \dots \\
&\leqslant e^{\sum_{k=n}^{n+m-1} A_k}\|x_n - p\| + e^{\sum_{k=n}^{n+m-1} A_k} \sum_{k=n}^{n+m-1} H_k \\
&\leqslant e^{\sum_{n=1}^{\infty} A_n}\|x_n - p\| + e^{\sum_{n=1}^{\infty} A_n} \sum_{k=n}^{n+m-1} H_k \\
&= Q\|x_n - p\| + Q \sum_{k=n}^{n+m-1} H_k \qquad (3.5)
\end{aligned}$$

where $Q = e^{\sum_{n=1}^{\infty} A_n}$.

Since $\liminf_{n\to\infty} d(x_n, F) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F$ such that $\|x_{n_k} - p_{n_k}\| \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\|x_{n_k} - p_{n_k}\| < \frac{\varepsilon}{4Q} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} H_k < \frac{\varepsilon}{4Q}, \tag{3.6}$$

for all $k \geqslant k_\varepsilon$.

For any $m \geqslant 1$ and for all $n \geqslant n_{k_\varepsilon}$, by (3.5), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - p_{n_k}\| + \|x_n - p_{n_k}\| \\ &\leqslant Q\|x_{n_k} - p_{n_k}\| + Q\sum_{k=n_{k_\varepsilon}}^{\infty} H_k \\ &\quad +Q\|x_{n_k} - p_{n_k}\| + Q\sum_{k=n_{k_\varepsilon}}^{\infty} H_k \\ &= 2Q\|x_{n_k} - p_{n_k}\| + 2Q\sum_{k=n_{k_\varepsilon}}^{\infty} H_k \\ &< 2Q.\frac{\varepsilon}{4Q} + 2Q.\frac{\varepsilon}{4Q} = \varepsilon. \end{aligned} \tag{3.7}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since C is a nonempty closed convex subset of Banach space E, so there exists a $q \in C$ such that $x_n \to q$ as $n \to \infty$. Finally, we prove that $q \in F$. In fact, notice that $d(q, F) = 0$. Therefore, for any $\varepsilon_1 > 0$, there exists a $p_2 \in F$ such that $\|p_2 - q\| < \varepsilon_1$. Then, we have

$$\begin{aligned} \|T_i q - q\| &\leqslant \|T_i q - p_2\| + \|p_2 - q\| \\ &\leqslant (L+1)\|p_2 - q\| < (L+1)\varepsilon_1. \end{aligned} \tag{3.8}$$

By the arbitrariness of $\varepsilon_1 > 0$, we have $T_i q = q$ for all $i = 1, 2, 3$, that is, q is a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

THEOREM 3.2. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.6) with the restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\{x_n\}$ converges to a common fixed point p of the mappings T_1, T_2 and T_3 if and only if there exists some infinite subsequence of $\{x_n\}$ which converges to p.*

PROOF. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. This completes the proof. □

As an application of our Theorem 3.1, we establish another strong convergence result as follows.

THEOREM 3.3. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.6) with the restrictions $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that the mappings T_1, T_2 and T_3 satisfy the following conditions:*

(C_1)

$$\lim_{n\to\infty}\|x_n - T_1x_n\| = 0, \lim_{n\to\infty}\|x_n - T_2x_n\| = 0, \lim_{n\to\infty}\|x_n - T_3x_n\| = 0;$$

(C_2) *there exists a constant* $A > 0$ *such that*

$$\Big\{\|x_n - T_1x_n\| + \|x_n - T_2x_n\| + \|x_n - T_3x_n\|\Big\} \geqslant Ad(x_n, F),\ \forall\, n \geqslant 1.$$

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* T_1, T_2 *and* T_3.

PROOF. From conditions (C_1) and (C_2), we have $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 3.1 implies that $\{x_n\}$ must converges strongly to a point of F. This completes the proof. □

THEOREM 3.4. *Let* E *be a real Banach space satisfying Opial's condition and* C *be a weakly compact subset of* E. *Let* $T_i\colon C \to C$, $(i = 1, 2, 3)$ *be uniformly* L*-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with* $\{r_n\}, \{s_n\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} r_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$. *Let* $\{x_n\}$ *be the sequence defined by (2.6) with the restrictions* $\sum_{n=1}^{\infty}\alpha_n < \infty$ *and* $\sum_{n=1}^{\infty}\lambda_n < \infty$. *Suppose that* T_1, T_2 *and* T_3 *have a common fixed point,* $I - T_i$ *for* $i = 1, 2, 3$ *is demiclosed at zero and* $\{x_n\}$ *is an approximating common fixed point sequence for* T_i *for* $i = 1, 2, 3$, *that is,* $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, *for* $i = 1, 2, 3$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of the mappings* T_1, T_2 *and* T_3.

PROOF. First, we show that $\omega_w(x_n) \subset F = \cap_{i=1}^{3} F(T_i)$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$ for $i = 1, 2, 3$. Since $I - T_i$ for $i = 1, 2, 3$ is demiclosed at zero, $x \in F = \cap_{i=1}^{3} F(T_i)$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

EXAMPLE 3.1. Let E be the real line with the usual norm $|.|$ and $K = [-1, 1]$. Define $T_1, T_2, T_3\colon K \to K$ by

$$T_1x = \text{sinx}, \quad x \in [-1, 1],$$
$$T_2x = x/3, \quad x \in [-1, 1],$$
$$T_3x = x/2, \quad x \in [-1, 1],$$

for $x \in K$. Obviously $T_1(0) = 0$, $T_2(0) = 0$ and $T_3(0) = 0$, that is, 0 is a common fixed point of T_1, T_2 and T_3, that is, $F = F(T_1) \cap F(T_2) \cap F(T_3) = \{0\}$. Thus T_1, T_2 and T_3 are quasi-nonexpansive mappings. It follows that T_1, T_2 and T_3 are asymptotically quasi-nonexpansive mappings with the constant sequence $\{1\}$. Also T_1, T_2 and T_3 are uniformly continuous on [-1,1]. Thus they are asymptotically quasi-nonexpansive mappings in the intermediate sense and hence are generalized asymptotically quasi-nonexpansive mappings by Remark 2.1.

REMARK 3.1. The main result of this paper can be extended to a finite family of generalized asymptotically quasi-nonexpansive mappings $\{T_i : 1 \leqslant i \leqslant N\}$ by introducing the following iteration scheme:

Let $T_1, T_2 \dots, T_N : C \to C$ be N generalized asymptotically quasi-nonexpansive mappings. Let $x_1 \in C$ be a given point. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_{n+1} &= (1 - a_{n_1} - b_{n_1})x_n + a_{n_1} T_1^n y_{n_1} + b_{n_1} u_{n_1}, \\
y_{n_1} &= (1 - a_{n_2} - b_{n_2})x_n + a_{n_2} T_2^n y_{n_2} + b_{n_2} u_{n_2}, \\
&\ \vdots \\
y_{n_{(N-2)}} &= (1 - a_{n_{(N-1)}} - b_{n_{(N-1)}})x_n + a_{n_{(N-1)}} T_{N-1}^n y_{n_{(N-1)}} \\
&\quad + b_{n_{(N-1)}} u_{n_{(N-1)}}, \\
y_{n_{(N-1)}} &= (1 - a_{n_N} - b_{n_N})x_n + a_{n_N} T_N^n x_n + b_{n_N} u_{n_N},\ n \geqslant 1,
\end{aligned}
\tag{3.9}
$$

is called N-step iterative sequence with errors of $T_1, T_2, \dots, T_N$, where $\{u_{n_i}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$, are N bounded sequences in C, and $\{a_{n_i}\}_{n=1}^{\infty}$, $\{b_{n_i}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$, are N appropriate sequences in $[0, 1]$.

Remark 3.2. Theorem 3.1 extends, improves and unifies the corresponding result of [**2, 8, 9, 12, 14, 15, 16**]. Especially Theorem 3.1 extends, improves and unifies Theorem 1 in [**8**], Theorem 1 and 2 in [**9**], Theorem 2.0.3 in [**15**] and Theorem 3.2 in [**16**] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [**8**], [**9**], [**15**] and [**16**] is extended to more general generalized asymptotically quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme in [**8**], the usual modified Ishikawa iteration scheme with errors in [**9**] and the usual modified Ishikawa iteration scheme with errors for two mappings in [**15**] and [**16**] are extended to the three-step iteration scheme with errors for three mappings.

Remark 3.3. Theorem 3.2 extends, improves and unifies Theorem 3 in [**9**] and Theorem 3.3 extends, improves and unifies Theorem 3 in [**8**] in the following aspects:

(1) The asymptotically quasi-nonexpansive mapping in [**8**] and [**9**] is extended to more general generalized asymptotically quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme in [**8**] and the usual modified Ishikawa iteration scheme with errors in [**9**] are extended to the three-step iteration scheme with errors for three mappings.

Remark 3.4. Our results also extend the corresponding results of Quan [**13**] to the case of more general class of uniformly quasi-Lipschitzian mapping considered in this paper.

Remark 3.5. Our results also extend the corresponding results of Xu and Noor [**18**] to the case of more general class of asymptotically nonexpansive mappings considered in this paper.

REMARK 3.6. Theorem 3.4 extends and improves Theorem 2.6 and 2.7 of Sahu and Jung [**14**] to the case of more general class of asymptotically quasi-nonexpansive type mappings and modified three-step iteration scheme with errors considered in this paper.

4. Conclusion

The class of generalized asymptotically quasi-nonexpansive mapping is more general than the class of asymptotically nonexpansive, asymptotically quasi-nonexp ansive and asymptotically nonexpansive mapping in the intermediate sense. Therefore, the results presented in this paper are improvement and generalization of several well-known results in the current literature (see, e.g., [**2, 8, 9, 12, 14, 15, 16, 18**]).

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