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THE TOTAL HUB NUMBER OF GRAPHS

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Abstract. Let G be a graph. A total hub set S of G is a subset of $V(G)$ such that every pair of vertices (whether adjacent or nonadjacent) of $V - S$ are connected by a path, whose all intermediate vertices are in S. The total hub number $h_t(G)$ is then defined to be the minimum cardinality of a total hub set of G. In this paper, the total hub number for several classes of graphs is computed, bounds in terms of other graph parameters are also determined.

1. Introduction

In this paper we are concerned with simple graphs, that have no loops and no multiple or directed edges. Let G be such a graph, and let p and q be the number of its vertices and edges, respectively. Then we say that G is an (p, q)-graph.

Consider the graphs that represent transportation networks, that is the vertices can be taken to be locations or destinations, and an edge exists between two vertices precisely when there is an "easy passage" between the corresponding locations. For example, a city's network of streets, with vertices representing intersections or other points of intersect, and edges road segments. We are connected with a certain kind of connectivity, specifically we want a set S such that any traffic between disparate points in our network passes solely through vertices in this set.

Suppose that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an S-path trivial.) A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - S$, there is an S-path in G between x and y. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$ [8].

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A subset S of a graph G is called a dominating set if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$ [2].

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers. We need the following to prove main results.

Theorem 1.1 ([8]). For any connected graph G, $h(G) \leqslant \gamma_c(G) \leqslant h(G) + 1$.

Theorem 1.2 ([3]). For any connected graph G, $\gamma_c(G) \leqslant p - \Delta(G)$.

Theorem 1.3 ([7]). For any graph G, $\lceil \frac{p}{\Delta(G)+1} \rceil \leqslant \gamma(G)$, where $\lceil x \rceil$ is a least integer not less than x.

Theorem 1.4 ([6]). For any connected graph G, $\gamma_c(G) \leqslant 2q - p$.

Theorem 1.5 ([7]). For any (p, q) graph G,

(1) $p - q \leqslant \gamma(G)$.
(2) $\lceil \frac{(d(G)+1)}{3} \rceil \leqslant \gamma(G)$.

2. The Total Hub Number of Graphs

Definition. Let G be a graph. A total hub set S of G is a subset of $V(G)$ such that every pair of vertices (whether adjacent or nonadjacent) of $V - S$ are connected by a path, whose all intermediate vertices are in S. The total hub number $h_t(G)$ is then defined to be the minimum cardinality of a total hub set of G.

It is clear that $h_t(G)$ is well-defined for any G, since $V(G)$ is a total hub set. In all situations of interest, we will assume G to be connected, if G is a disconnected graph then any total hub set must contain union of the set of vertices from all but one largest component, and the total hub set of the largest component.

It is obvious that any total hub set in a graph G is also a hub set, and thus we obtain the obvious bound $h(G) \leqslant h_t(G)$.

We now proceed to compute $h_t(G)$ for some standard graphs. It can be easily verified that

(1) For any path P_p with $p \geqslant 3$, $h_t(P_p) = p - 2$.
(2) For any cycle C_p,

$$h_t(C_p) = \begin{cases} p-2, & \text{if } p \leqslant 5 \ ; \\ p-3, & \text{if } p \geqslant 6 \ . \end{cases}$$

.

(3) For any tree T of order $p \geqslant 3$, $h_t(T) = p - s$, where s is the number of pendant vertices.
(4) For any complete graph K_p,

$$h_t(K_p) = \begin{cases} 0, & \text{if } p = 1 \ ; \\ 1, & \text{if } p \geqslant 2 \ . \end{cases}$$

(5) For the double star $S_{n,m}$, $n, m \geqslant 1$, $h_t(S_{n,m}) = 2$.

(6) For the complete bipartite graph $G \cong K_{n,m}$,

$$h_t(K_{n,m}) = \begin{cases} 1, & \text{if } n \text{ or } m = 1 \ ; \\ 2, & \text{if } n, m \geqslant 2 \ . \end{cases}$$

(7) For the wheel W_p denotes on p vertices, $h_t(W_p) = 1$.

Proposition 2.1. *For any connected graph G, $h(G) \leqslant h_t(G) \leqslant \gamma_c(G) \leqslant h(G) + 1$.*

Proof. Since $h(G) \leqslant h_t(G)$ and every connected dominating set of G is a total hub set, we have $h(G) \leqslant h_t(G) \leqslant \gamma_c(G)$. Also by Theorem 1.1, $\gamma_c(G) \leqslant h(G) + 1$. Hence $h(G) \leqslant h_t(G) \leqslant \gamma_c(G) \leqslant h(G) + 1$. □

A hub set of G need not be a total hub set of G. For example, in $C_5 = \{v_1, v_2, ..., v_5, v_1\}$, $\{v_1, v_2\}$ is a hub set but not a total hub set. Also a total hub set need not be a connected dominating set. For example, in $C_6 = \{v_1, v_2, ..., v_6, v_1\}$, $\{v_1, v_3, v_5\}$ is a total hub set but not a connected dominating set.

Proposition 2.2. *For any connected graph G, $\gamma(G) \leqslant h_t(G)$.*

Proof. Since every total hub set of G is a dominating set, we have $\gamma(G) \leqslant h_t(G)$. □

A dominating set of G need not be a total hub set of G. For example, in $C_5 = \{v_1, v_2, ..., v_5, v_1\}$, $\{v_1, v_3\}$ is a dominating set but not a total hub set.

For any disconnected graph G having k components $G_1, G_2, ..., G_k$ of orders $p_1, p_2, ..., p_k$ respectively, such that $p_1 \leqslant p_2 \leqslant ... \leqslant p_k$. We have $h_t(G) \leqslant p_1 + p_2 + ... + h_t(G_k)$.

Theorem 2.1. *For any connected (p, q) graph G with $p \geqslant 2$, $\Delta(G) = p - 1$, if and only if $h_t(G) = 1$.*

Proof. Let G be a connected graph such that $\Delta(G) = p - 1$. Suppose u is a vertex of G with $deg(u) = p-1$, then every pair of vertices of $V - \{u\}$ are connected by a path whose internal vertex is u. Therefore $h_t(G) = 1$.

Conversely, suppose $h_t(G) = 1$ and $S = \{u\}$ be a minimum total hub set of G. Since a total hub set is a dominating set, it follows that u is dominates all other vertices. Hence $\Delta(G) = p - 1$. □

Theorem 2.2. *For any connected graph G, $h_t(G) \leqslant p - \Delta(G)$.*

Proof. Proof follows by Proposition 2.1 and Theorem 1.2.

The bound is sharp for P_5. □

Corollary 2.1. *For any connected graph G with $p \geqslant 3$, $h_t(G) \leqslant p - 2$.*

Proof. Proof follows by Theorem 2.4. □

Theorem 2.3. *For any connected graph G with $p \geqslant 3$, $\lceil \frac{p}{\Delta(G)+1} \rceil \leqslant h_t(G) \leqslant 2q - p$.*

Proof. The lower bound follows by Theorem 1.3 and Proposition 2.2. The upper bound follows by Theorem 1.4 and Proposition 2.1. □

The lower bound is attained if $\Delta(G) = p - 1$ and the upper bound is attained for P_n, $n \geqslant 3$

Theorem 2.4. *For any (p, q) graph G with both G and $\overline{G}$ connected, $h_t(G) + h_t(\overline{G}) \leqslant p(p-3)$.*

Proof. By Theorem 2.6, $h_t(G) \leqslant 2q - p$ and $h_t(\overline{G}) \leqslant 2\overline{q} - p$, where $\overline{q}$ is the number of edges in $\overline{G}$. Therefore

$$\begin{aligned} h_t(G) + h_t(\overline{G}) &\leqslant 2(q + \overline{q}) - 2p \\ &= p(p-3) \end{aligned}$$

The bound is sharp for P_4. □

Theorem 2.5. *For any (p, q) graph G with both G and $\overline{G}$ connected, $h_t(G) + h_t(\overline{G}) \leqslant p + 1$.*

Proof. Form Theorem 2.4, $h_t(G) \leqslant p - \Delta(G)$ and $h_t(\overline{G}) \leqslant p - \Delta(\overline{G})$.
We have,

$$\begin{aligned} h_t(G) + h_t(\overline{G}) &\leqslant 2p - (\Delta(G) + \Delta(\overline{G})) \\ &= 2p - (\Delta(G) + p - 1 - \delta(G)) \\ &= p + 1 - (\Delta(G) - \delta(G)). \end{aligned}$$

Since $\Delta(G) - \delta(G) \geqslant 0$, we have $h_t(G) + h_t(\overline{G}) \leqslant p + 1$.
The bound is sharp for C_5. □

Theorem 2.6. *For any (p, q) graph G with both G and $\overline{G}$ connected,*

(1) $h_t(G) + h_t(\overline{G}) \leqslant 2(p-2)$.
(2) $h_t(G)h_t(\overline{G}) \leqslant (p-2)^2$.

Proof. Proof follows by Corollary 2.5.
The bound is sharp for P_4. □

Theorem 2.7. *For any (p, q) graph G,*

(1) $p - q \leqslant h_t(G)$.
(2) $\lceil \frac{(d(G)+1)}{3} \rceil \leqslant h_t(G)$.

Proof. Proof follows by Theorem 2.2 and Theorem 1.4. □

Theorem 2.8. *For any connected (p, q) graph G*
$q \leqslant \frac{1}{2}(p - h_t(G))(p + h_t(G) - 1)$.

Proof. Let G be a connected graph. If $h_t = 0$ then $G = K_1$ and we have $q = 0$ in the case. Now we prove by induction on p. If $p = 2$, K_2 is the only connected graph and we have $q = 1$ in this case. Thus the basis holds.

Assume that every graph G with vertices less than p and $h_t(G) \geqslant 1$ satisfies the inequality. Let G be a connected graph with p vertices. Suppose $h_t(G) \geqslant 2$. Let v

be a vertex of maximum degree $\Delta(G)$. By Theorem 2.4, $N(v) = \Delta(G) \leqslant p - h_t(G)$. Therefore $\Delta(G) = p - h_t(G) - r$, where $0 \leqslant r \leqslant p - h_t(G)$.

Let $S = V - N[v]$, then $|S| = h_t(G) + r - 1$. If $u \in N(v)$, then the set $(S - N(u)) \cup \{u, v\}$ is a total hub set of G. Therefore $h_t(G) \leqslant |S - N(u)| + 2$. Thus

$$\begin{aligned} h_t(G) &\leqslant |S| - |S \cap N(u)| + 2 \\ &= h_t(G) + r - 1 - |S \cap N(u)| + 2. \end{aligned}$$

Hence $|S \cap N(u)| \leqslant r + 1$ for each vertex $u \in N(v)$. Thus the number of edges between $N(v)$ and S, say m_1, is $m_1 \leqslant \Delta(G)(r + 1)$.

Furthermore, since G is connected, there exist two vertices $x \in S$ and $y \in N(v)$ such that x and y are adjacent, then there is a total hub path between v and x. Therefore, if H be a total hub set of the induced subgraph $\langle S\rangle$, then $H \cup N(v)$ is a total hub set of G. Hence $h_t(G) \leqslant |H \cup N(v)|$, it implies that, $h_t(\langle S\rangle) \geqslant h_t(G) - 3$. By the induction hypothesis, the number of edges in the induced subgraph $\langle S\rangle$, say m_2, is

$$\begin{aligned} m_2 &\leqslant \frac{1}{2}(|S| - h_t(\langle S\rangle))(|S| + h_t(\langle S\rangle) - 1) \\ &\leqslant \frac{1}{2}(r + 2)(2h_t(G) + r - 5) \\ &= \frac{1}{2}(r + 2)(2p - 2\Delta(G) - r - 5). \end{aligned}$$

Let $m_3 = |E(\langle N[v]\rangle)|$. Now vertex v is adjacent to $\Delta(G)$ edges and each vertex $u \in N(v)$ has degree at most $\Delta(G)$. The edges between S and $N(v)$ account for at most $r + 1$ edges incident to each $u \in N(v)$. Thus,

$$\begin{aligned} q &= m_1 + m_2 + m_3 \\ &\leqslant \Delta(G)(r + 1) + \frac{1}{2}(r + 2)(2p - r - 2\Delta(G) - 5) + \Delta(G) + \frac{1}{2}\Delta(G)(\Delta(G) - r - 2) \\ &= \Delta(G)(p - h_t(G) - \Delta(G) + 1) + \frac{1}{2}(p - h_t(G) - \Delta(G) + 2)(p + h_t(G) - \Delta(G) - 5) \\ &+ 2\Delta(G) + \frac{1}{2}\Delta(G)(2\Delta(G) - p + h_t(G) - 2) \\ &= \frac{1}{2}(p - h_t(G))(p + h_t(G) - 1) - (p - 3h_t(G) + 5 + \frac{1}{2}\Delta(G)(p + h_t(G) - \Delta(G) - 5) \\ &\leqslant \frac{1}{2}(p - h_t(G))(p + h_t(G) - 1). \end{aligned}$$

Hence $q \leqslant \frac{1}{2}(p - h_t(G))(p + h_t(G) - 1)$.
Hence result holds for graphs having $h_t(G) \geqslant 2$.
Suppose graph G is having $h_t(G) \geqslant 1$. Add an isolated vertex to G. Thus G is a graph with $p + 1$ vertices, q edges and the maximum degree $\Delta(G)$. Thus the result also holds for graphs having $h_t(G) \geqslant 1$.

The equality holds for a complete graph. □

Corollary 2.2. *For any connected (p, q) graph G, $h_t(G) \leqslant \sqrt{p^2 - 2q}$.*

Proof. By Theorem 2.11, we have

$$\begin{aligned} 2q &\leqslant (p - h_t(G))(p + h_t(G) - 1) \\ &= p^2 - p - h_t^2(G) + h_t(G). \end{aligned}$$

Since $p - h_t(G) \geqslant 1$, we have $2q < p^2 - h_t^2(G)$. Hence $h_t(G) < \sqrt{p^2 - 2q}$. □

Corollary 2.3. *For any (p, q) graph G with both G and $\overline{G}$ connected, $h_t^2(G) + h_t^2(\overline{G}) \leqslant p(p + 1)$.*

Proof. By Corollary 2.12, $h_t^2(G) \leqslant p^2 - 2q$ and $h_t^2(\overline{G}) \leqslant p^2 - 2\overline{q}$, where $\overline{q}$ is the number of edges in $\overline{G}$.
Thus,

$$\begin{aligned} h_t^2(G) + h_t^2(\overline{G}) &\leqslant 2p^2 - 2(q + \overline{q}) \\ &= 2p^2 - p(p - 1) \\ &= p^2 + p \\ &= p(p + 1). \end{aligned}$$

□

Theorem 2.9. *For any connected (p, q) graph G, $q \leqslant \frac{1}{2}(p + h_t(G))(p + h_t(G) - 1)$.*

Proof. Since $(p - h_t(G)) \leqslant (p + h_t(G))$, the result follows by Theorem 2.11. □

Corollary 2.4. *For any connected (p, q) graph G, $\sqrt{2q} - p < h_t(G)$.*

Proof. By Theorem 2.14, we have

$$\begin{aligned} 2q &\leqslant (p + h_t(G))(p + h_t(G) - 1) \\ &= p^2 + h_t^2(G) + 2h_t(G)p - (p + h_t(G)). \end{aligned}$$

Since $(p + h_t(G)) > 1$, we have $2q < (p + h_t(G))^2$.
Hence $\sqrt{2q} - p < h_t(G)$. □

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