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AN UPPER BOUND TO THE SECOND HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. The objective of this paper is to introduce certain subclass of analytic functions and obtain an upper bound to the second Hankel determinant $|a_2a_4 - a_3^2|$ for the function f, belonging to this class, using Toeplitz determinants.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

The Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$ was defined by Pommerenke [27, 28] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by many authors in the literature [23]. For example, Noor [24] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the

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functions in S with a bounded boundary. Ehrenborg [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [14]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order α $(0 < \alpha \leqslant 1)$ denoted by $\widetilde{ST}(\alpha)$. In this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus [13] have considered the functional$|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [17]. In their work, they have shown that if $f \in$ RT then $|a_2a_4 - a_3^2| \leqslant \frac{4}{9}$. The same authors [12] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex functions denoted by ST and CV of S and have shown that $|a_2a_4 - a_3^2| \leqslant 1$ and $|a_2a_4 - a_3^2| \leqslant \frac{1}{8}$ respectively. Mishra and Gochhayat [20] have obtained sharp bound to the non- linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leqslant \rho \leqslant 1, 0 \leqslant \lambda < 1, |\alpha| < \frac{\pi}{2})$, defined as $Re\left\{e^{i\alpha}\frac{\Omega_z^\lambda f(z)}{z}\right\} > \rho\cos\alpha$, using the fractional differential operator denoted by Ω_z^λ, defined by Owa and Srivastava [24]. These authors have shown that, if $f \in R_\lambda(\alpha, \rho)$ then

$$|a_2a_4 - a_3^2| \leqslant \left\{\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2 cos^2\alpha}{9}\right\}.$$

Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors $([1], [3, 4], [9 - 11], [18, 19], [21, 22], [29], [31 - 39])$.

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we introduce certain subclass of analytic functions and obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f belonging to this class , defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be in the class RST$(\beta)(\beta \geqslant 0)$ in E, if it satisfies the condition

$$Re\left\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta f'(z)\right\} \geqslant 0, \ \forall z \in E. \tag{1.4}$$

It is observed that for $\beta = 0$ and $\beta = 1$ in (1.4), we respectively get $RST(0) = ST$ and $RST(1) = RT$.

2. Preliminary Results

Let P denote the class of functions p analytic with $\text{Re}\{p(z)\} > 0, \forall z \in E$. Some preliminary Lemmas required for proving our result are as follows:

$$\text{(2.1)} \qquad \text{Where } p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \forall z \in E.$$

Here $p(z)$ is called as Carathéodory function [5].

LEMMA 2.1 ([**26**], [**30**]). *If $p \in \mathrm{P}$, then $|c_k| \leqslant 2$, for each $k \geqslant 1$ and the inequality is sharp.*

LEMMA 2.2 ([**8**]). *The power series for p given in (2.1) converges in the unit disc E to a function in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = \; 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geqslant m$.

This necessary and sufficient condition found in [8] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \geqslant 0,$$

which is equivalent to

$$\text{(2.2)} \qquad 2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, \text{ with} |x| \leqslant 1.$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geqslant 0$ is equivalent to

$$\text{(2.3)} \qquad |(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leqslant 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2.$$

From the relations (2.2) and (2.3), after simplifying, we get

$$\text{(2.4)} \qquad 4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}$$

$$\text{for some real value of } z, \text{ with } |z| \leqslant 1.$$

To obtain our result, we referred to the method by Libera and Zlotkiewicz [15, 16].

3. Main Result

THEOREM 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in RST(\beta)$ $(0 \leqslant \beta \leqslant 1)$ *then*

$$|a_2a_4 - a_3^2| \leqslant \left[\frac{2}{(2+\beta)}\right]^2$$

and the result is sharp.

PROOF. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in RST(\beta)$, from the Definition 1.1, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$ such that

$$\left\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta f'(z)\right\} = p(z) \tag{3.1}$$

$$\Leftrightarrow [\{(1-\beta)z + \beta f(z)\} \times f'(z)] = [f(z) \times p(z)].$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in the relation (3.1), we have

$$\left[(1-\beta)z\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} + \beta\left\{z + \sum_{n=2}^{\infty} a_n z^n\right\} \times \left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\}\right] = \left[\left\{z + \sum_{n=2}^{\infty} a_n z^n\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right]. \tag{3.2}$$

Upon simplification, we obtain

$$\left[1 + (1+\beta)a_2 z + \left\{(2+\beta)a_3 + 2\beta a_2^2\right\} z^2 + \{(3+\beta)a_4 + 5\beta a_2 a_3\} z^3 + ...\right] = \left[1 + c_1 z + (c_2 + c_1 a_2)z^2 + (c_3 + c_2 a_2 + c_1 a_3)z^3 + ...\right]. \tag{3.3}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.3), after simplifying, we get

$$[a_2 = \frac{c_1}{(1+\beta)};\ a_3 = \frac{1}{(1+\beta)^2(2+\beta)}\left\{(1+\beta)^2 c_2 + (1-\beta)c_1^2\right\}; \quad a_4 = \frac{1}{(1+\beta)^3(2+\beta)(3+\beta)} \times \left\{(1-\beta)(1-4\beta)c_1^3 + 3(1-\beta)(1+\beta)^2 c_1 c_2 + (1+\beta)^3(2+\beta)c_3\right\}]. \tag{3.4}$$

Substituting the values of a_2, a_3 and a_4 from the relation (3.4) in the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in RST(\beta)$, we have

$$|a_2a_4 - a_3^2| = \left|\frac{c_1}{(1+\beta)} \times \frac{1}{(1+\beta)^3(2+\beta)(3+\beta)} \times \left\{(1-\beta)(1-4\beta)c_1^3 + 3(1-\beta)(1+\beta)^2 c_1 c_2 + (1+\beta)^3(2+\beta)c_3\right\} - \frac{1}{(1+\beta)^4(2+\beta)^2}\left\{(1+\beta)^2 c_2 + (1-\beta)c_1^2\right\}^2\right|.$$

Upon simplification, we obtain

$$(3.5)\quad |a_2a_4 - a_3^2| = \frac{1}{(1+\beta)^4(2+\beta)^2(3+\beta)} \times |(1+\beta)^3(2+\beta)^2c_1c_3 + \beta(1-\beta)(1+\beta)^2c_1^2c_2 - (1+\beta)^4(3+\beta)c_2^2 - (1-\beta)(3\beta^2+5\beta+1)c_1^4|.$$

The expression (3.5) is equivalent to

$$(3.6)\quad |a_2a_4 - a_3^2| = \frac{1}{(1+\beta)^4(2+\beta)^2(3+\beta)} \times \left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|.$$

$$(3.7)\quad \text{Where } \{d_1 = (1+\beta)^3(2+\beta)^2;\ d_2 = \beta(1-\beta)(1+\beta)^2;\ d_3 = -(1+\beta)^4(3+\beta);\ d_4 = -(1-\beta)(3\beta^2+5\beta+1)\}.$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 in the right hand side of (3.6), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4|.$$

Using the facts $|pa + qb| \leqslant |p||a| + |q||b|$, where p, q, a and b are real numbers and $|z| < 1$ in the right side the above expression, after simplifying, we get

$$(3.8)\quad 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2) + \left\{2(d_1+d_2+d_3)c_1^2|x| - \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}|x|^2\right\} \times (4-c_1^2)|.$$

Using the values of $d_1,\ d_2,\ d_3$ and d_4 from (3.7), upon simplification, we obtain

$$(3.9)\quad \{(d_1 + 2d_2 + d_3 + 4d_4) = (-2\beta^4 + 11\beta^3 + 13\beta^2 - 11\beta - 3);\ d_1 = (1+\beta)^3(2+\beta)^2;\ (d_1+d_2+d_3) = (1+\beta)^2(-\beta^2+2\beta+1)\}.$$

$$(3.10)\quad \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} = (1+\beta)^3\left\{c_1^2 + 2(2+\beta)^2c_1 + 4(1+\beta)(3+\beta)\right\}.$$

Consider

$$\begin{aligned}(3.11)\quad & \left\{c_1^2 + 2(2+\beta)^2c_1 + 4(1+\beta)(3+\beta)\right\} \\ &= \left[\left\{c_1 + (2+\beta)^2\right\}^2 - (2+\beta)^4 + 4(1+\beta)(3+\beta)\right]. \\ &= \left[\left\{c_1 + (2+\beta)^2\right\}^2 - \left\{\sqrt{\beta^4 + 8\beta^3 + 20\beta^2 + 16\beta + 4}\right\}^2\right] \\ &= \left[c_1 + \left\{(2+\beta)^2 + \sqrt{\beta^4 + 8\beta^3 + 20\beta^2 + 16\beta + 4}\right\}\right] \times \\ &\quad \left[c_1 + \left\{(2+\beta)^2 - \sqrt{\beta^4 + 8\beta^3 + 20\beta^2 + 16\beta + 4}\right\}\right].\end{aligned}$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geqslant (c_1-a)(c_1-b)$, where $a, b \geqslant 0$ in the right hand side of (3.11), after simplifying, we get

$$\left\{c_1^2 + 2(2+\beta)^2 c_1 + 4(1+\beta)(3+\beta)\right\} \geqslant \left\{c_1^2 - 2(2+\beta)^2 c_1 + 4(1+\beta)(3+\beta)\right\} \tag{3.12}$$

From the relations (3.10) and (3.12), we obtain

$$-\left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} \leqslant -(1+\beta)^3\left\{c_1^2 - 2(2+\beta)^2 c_1 + 4(1+\beta)(3+\beta)\right\}. \tag{3.13}$$

Substituting the calculated values from (3.9) and (3.13) in the right hand side of (3.8), we have

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leqslant |(-2\beta^4 + 11\beta^3 + 13\beta^2 - 11\beta - 3)c_1^4 + \\ & 2(1+\beta)^3(2+\beta)^2c_1(4-c_1^2) + 2(1+\beta)^2(-\beta^2+2\beta+1)c_1^2(4-c_1^2)|x| \\ & - (1+\beta)^3\left\{c_1^2 - 2(2+\beta)^2c_1 + 4(1+\beta)(3+\beta)\right\}(4-c_1^2)|x|^2|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying Triangle inequality and replacing $\mid x \mid$ by μ in the right hand side of the above inequality, we have

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leqslant [(2\beta^4 + 11\beta^3 + 13\beta^2 + 11\beta + 3)c^4 + \\ & 2(1+\beta)^3(2+\beta)^2c(4-c^2) + 2(1+\beta)^2(-\beta^2+2\beta+1)c^2(4-c^2)\mu \\ & + (1+\beta)^3\left\{c^2 - 2(2+\beta)^2c + 4(1+\beta)(3+\beta)\right\}(4-c^2)\mu^2]. \\ &= F(c,\mu), \quad \text{for} \quad 0 \leqslant \mu = |x| \leqslant 1. \end{aligned} \tag{3.14}$$

$$\begin{aligned} \text{Where } F(c,\mu) &= [(2\beta^4 + 11\beta^3 + 13\beta^2 + 11\beta + 3)c^4 + \\ & 2(1+\beta)^3(2+\beta)^2c(4-c^2) + 2(1+\beta)^2(-\beta^2+2\beta+1)c^2(4-c^2)\mu \\ & + (1+\beta)^3\left\{c^2 - 2(2+\beta)^2c + 4(1+\beta)(3+\beta)\right\}(4-c^2)\mu^2]. \end{aligned} \tag{3.15}$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,1] \times [0,2]$. Differentiating $F(c,\mu)$ in (3.15) partially with respect to μ, we get

$$\begin{aligned} \frac{\partial F}{\partial \mu} &= [2(1+\beta)^2(-\beta^2+2\beta+1)c^2(4-c^2)c^2 \\ & + 2(1+\beta)^3\left\{c^2 - 2(2+\beta)^2c + 4(1+\beta)(3+\beta)\right\}(4-c^2)\mu]. \end{aligned} \tag{3.16}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $o \leqslant \beta \leqslant 1$, from (3.16), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value at any point in the interior of the closed square $[0,1] \times [0,2]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0 \leqslant \mu \leqslant 1} F(c,\mu) = F(c,1) = G(c). \tag{3.17}$$

From the relations (3.15) and (3.17), upon simplification, we obtain

$$G(c) = F(c,1) = \left\{2\beta^2(2\beta^2+5\beta+1)c^4 - 4\beta(1+\beta)^2(\beta^2+7\beta+2)c^2 + 16(1+\beta)^4(3+\beta)\right\}. \tag{3.18}$$

$$G'(c) = \left\{8\beta^2(2\beta^2+5\beta+1)c^3 - 8\beta(1+\beta)^2(\beta^2+7\beta+2)c\right\}. \tag{3.19}$$

$$G''(c) = \left\{24\beta^2(2\beta^2+5\beta+1)c^2 - 8\beta(1+\beta)^2(\beta^2+7\beta+2)\right\}. \tag{3.20}$$

For Optimum value of G(c), consider $G'(c) = 0$. From (3.19), we get

$$8\beta c\left\{\beta(2\beta^2+5\beta+1)c^2 - (1+\beta)^2(\beta^2+7\beta+2)\right\} = 0. \tag{3.21}$$

We now discuss the following Cases.

Case 1) If $c \neq 0$ and $\beta = 0$, then, we have $G'(c) = 0$ and $G''(c) = 0$. Therefore, $G(c)$ is constant and the constant value is 48. i.e., $G(c) = 48$.

Case 2) If $c = 0$ and $\beta = 0$, then, we have $G'(c) = 0$ and $G''(c) = 0$.
In this Case also, we get $G(c) = 48$, which is a constant.
From Cases 1 and 2, we conclude that $G(c) = 48$, a constant, for every $c \in [0,2]$, provided $\beta = 0$.

Case 3) If $c = 0$ and $\beta \neq 0$, then, we have $G'(c) = 0$ and
$G''(c) = -8\beta(1+\beta)^2(\beta^2+7\beta+2) < 0, \quad \text{for} \quad 0 < \beta \leqslant 1.$

By the second derivative test, $G(c)$ has maximum value at $c = 0$.

Case 4) If $c \neq 0$ and $\beta \neq 0$, from (3.21), on using the Division algorithm for polynomials, we obtain

$$c^2 = \left[\frac{1}{4}\left\{(2\beta+13) + \frac{(\beta^2+31\beta+8)}{(2\beta^3+5\beta^2+\beta)}\right\}\right] > 0, \text{for } 0 < \beta \leqslant 1. \tag{3.22}$$

Substituting the value of c^2 from (3.22) in (3.20), after simplifying, we get

$$G''(c) = (16\beta^5 + 144\beta^4 + 272\beta^3 + 172\beta^2 + 32\beta) > 0, \text{ for } 0 < \beta \leqslant 1.$$

Therefore, by the second derivative test $G(c)$ has minimum value at c, where c^2 is given by (3.22). From the expression (3.18), the maximum value of $G(c)$ at $c = 0$ is given by

$$G_{max} = G(0) = 16(1+\beta)^4(3+\beta). \tag{3.23}$$

Considering maximum value of $G(c)$ at $c = 0$ only, from the relations (3.14) and (3.23), after simplifying, we get

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant 4(1+\beta)^4(3+\beta). \tag{3.24}$$

From the expressions (3.6) and (3.24), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \leqslant \left[\frac{2}{(2+\beta)}\right]^2. \tag{3.25}$$

Choosing $c_1 = c = 0$ and selecting $x = -1$ in (2.2) and (2.4), we find that $c_2 = -2$ and $c_3 = 0$. Substituting these values in (3.24), we see that equality is attained which shows that our result is sharp. This completes the proof of our Theorem. □

REMARK 3.1. Choosing $\beta = 0$, we have $RST(0) = ST$, from (3.25), we obtain $|a_2a_4 - a_3^2| \leqslant 1$ and this inequality is sharp.

REMARK 3.2. For the choice of $\beta = 1$, we have $RST(1) = RT$, for which, from (3.25), we get $|a_2a_4 - a_3^2| \leqslant \frac{4}{9}$ and is sharp.

Both the results coincide with those of Jateng, Halim and Darus ([12],[13]).

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