Module Extraction in Ontologies: The Case Of A Single Concept Name

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Abstract—**Modularity is important in the collaborative development of large scale ontologies. The subject has already been studied deeply and different notions of modularity have been introduced. Further different algorithms for extracting module are developed. Such an extraction requires an input ontology and a signature. The extracted module is a subset of the original ontology containing no concept name and role names other than those occurring in the input signature. In this paper we focus on a very special case of importing a single concept i.e., our input signature contains only a single concept name. We will see that we always end up with the same set of modules (modulo the concept name being imported). An extracted module for a single concept name (using the current techniques), may contain a lot more other concept names and role names, we prove that when we are only interested in the consequences of the extracted module, we don't get more information except of what we get from the set of modules we present here.**

Index Terms—**Module Extraction, Ontologies, Description Logics.**

I. INTRODUCTION

WITH the increased popularity and growth in size of ontologies, a natural question that arises is the one of the re-usability. The ideas of modularity and re-usability from software engineering guided people to study such notions in ontologies. Modularity is not only important when dealing with re-usability but we also need it in collaborative development of ontologies. Because of the expressiveness and semantic of the underlying ontology language, module extraction is rather a complex task. An important question is how to define modularity in ontologies. We come accross some notions of modularity in ([2], [3], [4], [5]). We will use the existing notion of modularity from [3].

In this paper, we focus on a very special case. We study the import of a single concept name, say A. This issue is already studied in [5] very deeply, where an algorithm for importing a single concept name has been presented. The output module contains additional concept names and roles names. In [2], we come up with another algorithm for module extraction, but a general one in the sense that we are not restricted to the import of a single concept name. Yet the resulting module may contain many other symbols as well. In literature, a module in an ontology is required to be the subset of the ontology. We will drop this condition for our case. We will see that when importing a single concept name A, there is always a TBox[1] in the set $\Pi_A = \{\emptyset, \{A = \top\}, \{A = \bot\}, \{\neg(\top \sqsubseteq A)\}, \{\neg(A \sqsubseteq$

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[1] We will be using the terms *module* and *TBox* interchangeably

$\bot)\}, \{\neg(\top \sqsubseteq A), \neg(A \sqsubseteq \bot)\}, \{\bot \sqsubseteq \top\}\}$ which serves as our required module.

We start by providing some basic definitions and notions in Section II. The actual result is presented in Section III. We present the notion of modularity in Definition III.1. Further, we present Theorem III.3 regarding the import of single concept name and later on prove it. The paper is concluded in Section V, where we will discuss some of the practical aspects of the import of a single concept name and complexity issues.

II. PRELIMINARIES

We introduce the description logic $\mathcal{ALC}$ where $\mathcal{ALC}$ stands for "attributive language with complement".

A. Syntax

A *signature* of $\mathcal{ALC}$ is (disjoint) union of a set N_C of concept names and a set N_R of role names. Let Σ be an $\mathcal{ALC}$ signature. The set of $\mathcal{ALC}$ *concept descriptions* (over Σ) is inductively defined as follows:

- every concept name in Σ is an $\mathcal{ALC}$ concept description.
- $\top$ (top concept) and $\bot$ (bottom concept) are $\mathcal{ALC}$ concept descriptions.
- If C, D are $\mathcal{ALC}$ concept descriptions over Σ and r is a role name in Σ then the following are concept descriptions:
  - $C \sqcap D$ (conjunction)
  - $C \sqcup D$ (disjunction)
  - $\neg C$ (negation)
  - $\exists r.C$ (existential restriction)
  - $\forall r.C$ (value restriction)

  Concept names, $\top$, and $\bot$ are called *atomic* and all the other descriptions are called *complex*.

  A *terminological axiom* is of the form

  $$C \sqsubseteq D$$

  where C, D are concept descriptions. Such axioms are called *General Concept Inclusions (GCIs)*.

  We extend the usual definition of a TBox by allowing negation of such inclusions as well. A *TBox* is a finite set of GCIs or their negations, where the negation of a GCI $C \sqsubseteq D$ is denoted by $\neg(C \sqsubseteq D)$.

  For a concept description C, we define signature of C (denoted by $\mathrm{Sig}(\{C\})$) as the set of all concept names and role names occurring in C. Similarly, for a TBox $\mathcal{T}$, we define the signature of $\mathcal{T}$ (denoted by $\mathrm{Sig}(\mathcal{T})$) as the set of all concept names and role names occurring in $\mathcal{T}$. Further, we extend the definition of Sig as follows:

For $X_1, X_2, ..., X_n$ $(n \geq 0)$:

$$\text{Sig}(X_1 \cup X_2 \cup ... \cup X_n) := \bigcup_{1 \leq i \leq n} \text{Sig}(X_i)$$

where the X_i's are TBoxes or singleton sets of the form $\{C\}$ with C being a concept description.

B. Semantics

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, the interpretation domain, and an *extension mapping* $\cdot^{\mathcal{I}}$, that maps;
- $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
- $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

The mapping $\cdot^{\mathcal{I}}$ is extended to the other concept descriptions as follows:
- $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}, \bot^{\mathcal{I}} := \emptyset$
- $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}, (\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
- $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} | \exists e \in \Delta^{\mathcal{I}} \text{ with } (d, e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\}$
- $(\forall r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} | \forall e \in \Delta^{\mathcal{I}}, (d, e) \in r^{\mathcal{I}} \text{ implies } e \in C^{\mathcal{I}}\}$

An interpretation is said to be *finite* if it's domain is finite. An interpretation $\mathcal{I}$ *satisfies* a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and if not then its negation is satisfied. $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, in symbol $\mathcal{I} \models \mathcal{T}$, if it satisfies every GCI or negated GCI in $\mathcal{T}$. If $\mathcal{I}$ is a finite interpretation and is a model of a TBox $\mathcal{T}$, then we say that $\mathcal{I}$ is a *finite model* of $\mathcal{T}$.

A concept C is *satisfiable* with respect to $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}}$ is nonempty. In this case we also say that $\mathcal{I}$ is a model of C.

A concept C is *subsumed* by a concept D with respect to $\mathcal{T}$, $\mathcal{T} \models C \sqsubseteq D$ in symbol, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.

A TBox $\mathcal{T}$ entails a TBox $\mathcal{T}'$ (in symbol $\mathcal{T} \models \mathcal{T}'$) if for all interpretation $\mathcal{I}$, we have that $\mathcal{I} \models \mathcal{T}$ implies $\mathcal{I} \models \mathcal{T}'$.

Definition II.1 (ω-Lift of an Interpretation). For any finite interpretation $\mathcal{I}$, the *lifting* of $\mathcal{I}$ to ω is the interpretation $\mathcal{I}_\omega$ defined as follows:
- $\Delta^{\mathcal{I}_\omega} := \{(x, i) | x \in \Delta^{\mathcal{I}} \wedge i \in \mathbb{N}\}$
- $A^{\mathcal{I}_\omega} := \{(a, i) | a \in A^{\mathcal{I}} \wedge i \in \mathbb{N}\}$ for every concept name $A \in N_C$
- $r^{\mathcal{I}_\omega} := \{(\langle a, i \rangle, \langle a', i \rangle) | (a, a') \in r^{\mathcal{I}} \wedge i \in \mathbb{N}\}$ for role name $r \in N_R$

From now onward, all the interpretations we will be considering are finite except for the interpretations obtained by lifting some finite interpretation to ω.

Lemma II.2. *Let C be an $\mathcal{ALC}$-concept description satisfiable w.r.t. a TBox $\mathcal{T}$ and $\mathcal{I}$ a model of $\mathcal{T}$ and C. Then $\mathcal{I}_\omega$ is also a model of $\mathcal{T}$ and C.*

Proof: We show that for all $(d, i) \in \Delta^{\mathcal{I}_\omega}$ and all concept names D, $(d, i) \in D^{\mathcal{I}_\omega}$ iff $d \in D^{\mathcal{I}_\omega}$. The proof is by the induction on the structure of D.
- For the atomic concepts, it follows from the definition of $\mathcal{I}_\omega$.

- Let $D = D_1 \sqcap D_2$ and $d \in \Delta^{\mathcal{I}}$:
  $d \in D^{\mathcal{I}}$
  $\Leftrightarrow d \in D_1^{\mathcal{I}}$ and $d \in D_2^{\mathcal{I}}$
  $\Leftrightarrow (d, i) \in D_1^{\mathcal{I}_\omega}$ and $(d, i) \in D_2^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (induction hypothesis)
  $\Leftrightarrow (d, i) \in D^{\mathcal{I}_\omega}$.
  The case $D = \neg D'$ can be treated in a similar fashion.
- Now suppose $D = \exists r D'$ and $d \in \Delta^{\mathcal{I}}$:
  $d \in D^{\mathcal{I}}$
  $\Leftrightarrow$ there is a $d' \in \Delta^{\mathcal{I}}$ with $(d, d') \in r^{\mathcal{I}}$ and $d' \in D'^{\mathcal{I}}$
  $\Leftrightarrow (d', i) \in \Delta^{\mathcal{I}_\omega}$ with $\langle (d, i), (d', i) \rangle \in r^{\mathcal{I}_\omega}$ and $(d', i) \in D'^{\mathcal{I}_\omega}$ for $i \in \mathbb{N} \Leftrightarrow d \in D^{\mathcal{I}_\omega}$.

Further, it follows from the prove that for a GCI $C \sqsubseteq D$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ iff $C^{\mathcal{I}_\omega} \subseteq D^{\mathcal{I}_\omega}$. Hence $\mathcal{I}$ satisfies $C \sqsubseteq D$ iff $\mathcal{I}_\omega$ does so. Further $\mathcal{I}_\omega \models \mathcal{T}$ iff $\mathcal{I} \models \mathcal{T}$.
∎

III. MODULARITY

In [3], a module is defined as follows:

Definition III.1 (Module). Let $\mathcal{T}^* \subseteq \mathcal{T}$ be ontologies and S a signature. We say that $\mathcal{T}^*$ is a *S-module* in $\mathcal{T}$, if for every ontology $\mathcal{T}'$ and concept descriptions C and D with $\text{Sig}(\mathcal{T}' \cup \{C\} \cup \{D\}) \cap \text{Sig}(\mathcal{T}) \subseteq S$, we have $\mathcal{T}' \cup \mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}' \cup \mathcal{T}^* \models C \sqsubseteq D$.

In our case, we are interested in the import of a single concept name A i.e., $S = \{A\}$. As already mentioned, we drop the condition of a module $\mathcal{T}^*$ being a subset of the (original) TBox $\mathcal{T}$. We will show that there is a fixed set of TBoxes such that for any signature S={A}, one of the TBoxes satisfies the requirements of the module given in Definition III.1 except that it needs not be a subset of the original TBox $\mathcal{T}$. This makes sense because we are interested in the consequences of a TBox regardless of its being a subset or not. Hence we introduce the notion of a *signature substitute*.

Definition III.2 (Σ-Substitute). Let $\mathcal{T}$ and $\mathcal{T}^*$ be two TBoxes and Σ a signature. $\mathcal{T}^*$ is called Σ-*substitute* for $\mathcal{T}$ if for all TBoxes $\mathcal{T}'$ with $\text{Sig}(\mathcal{T}) \cap \text{Sig}(\mathcal{T}') \subseteq \Sigma$, we have:
$$\mathcal{T}' \cup \mathcal{T} \models C \sqsubseteq D \text{ iff } \mathcal{T}' \cup \mathcal{T}^* \models C \sqsubseteq D$$
for any $\mathcal{ALC}$ concept description C and D over Σ.

Based on the notion presented in Definition III.2, we state the following theorem.

Theorem III.3. *Let $\mathcal{T}$ be a TBox and A a concept name with $A \in Sig(\mathcal{T})$. Then there is a $\mathcal{T}^* \in \Pi_A$, where*

$$\Pi_A = \{\emptyset, \{A = \top\}, \{A = \bot\}, \{\neg(\top \sqsubseteq A)\}, \{\neg(A \sqsubseteq \bot)\}, \{\neg(\top \sqsubseteq A), \neg(A \sqsubseteq \bot)\}, \{\top \sqsubseteq \bot\}\}$$

such that $\mathcal{T}^*$ is a $\{A\}$-substitute for $\mathcal{T}$.

Note that $\mathcal{T}^*$ satisfies the conditions of being a module of $\mathcal{T}$ except that we don't require for $\mathcal{T}^*$ to be a subset of $\mathcal{T}$. As we are assuming that any importing TBox $\mathcal{T}'$ shares only the symbol A with original TBox $\mathcal{T}$, the meaning of A in $\mathcal{T}$ can be as one of the following:

- $\mathcal{T}$ says nothing about A.
- A is the whole domain.
- A is an unsatisfiable concept.
- A is not representing the whole domain.
- A is not an unsatisfiable concept.
- TBox $\mathcal{T}$ is unsatisfiable.

In Theorem III.3, Π_A is the set of TBoxes where each TBox represents exactly one of the above mentioned cases. These TBoxes are related with one another in the sense that $(\Pi_A, \models)$ forms the lattice given in Figure 1. Note that there is always a TBox in Π_A, namely $\emptyset$, which is entailed by the original TBox $\mathcal{T}$. But there may be other TBoxes $\mathcal{T}^* \in \Pi_A$ such that $\mathcal{T} \models \mathcal{T}^*$. Just as an example, consider the case where $\mathcal{T} \models \{\top \sqsubseteq \bot\}$, then of course $\mathcal{T} \models \mathcal{T}^*$ for all $\mathcal{T}^* \in \Pi_A$. By observing the structure of the lattice we constitute the following lemma.
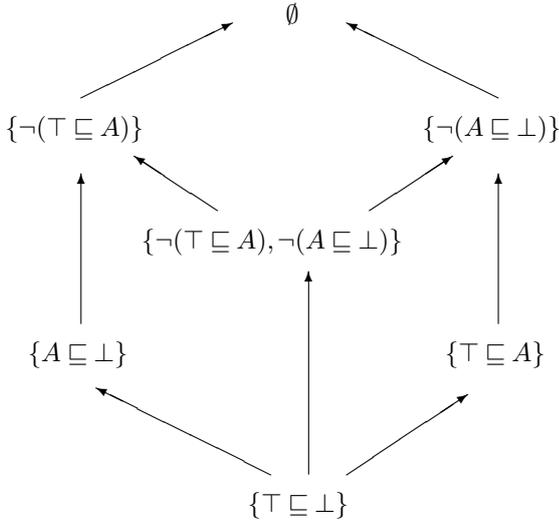

Fig. 1. Lattice induced by $(\Pi_A, \models)$

Lemma III.4. *Let $\mathcal{T}$ be any TBox and A a concept name. Then there is a unique $\mathcal{T}^* \in \Pi_A$ with $\mathcal{T} \models \mathcal{T}^*$ such that for all $\mathcal{T}^{**} \in \Pi_A$: if $\mathcal{T} \models \mathcal{T}^{**}$ then $\mathcal{T}^* \models \mathcal{T}^{**}$.*

Proof: The proof of the lemma is just a consequence of the induction of the lattice by Π_A with respect to $\models$. The existence of such a unique TBox $\mathcal{T}^*$ follows from the construction of the lattice i.e., for all TBoxes $\mathcal{T}^{**} \in \Pi_A$ with $\mathcal{T} \models \mathcal{T}^{**}$, $\mathcal{T}^*$ is the most specific of these TBoxes. ∎

Proof: **(Theorem III.3)**
Let $\mathcal{T}$ and A be as in Theorem III.3. By Lemma III.4, there is a unique $\mathcal{T}^* \in \Pi_A$ with $\mathcal{T} \models \mathcal{T}^*$ such that for all $\mathcal{T}^{**} \in \Pi_A$ we have: if $\mathcal{T} \models \mathcal{T}^*$ then $\mathcal{T} \models \mathcal{T}^{**}$. We show $\mathcal{T}^*$ is a Σ-substitute for $\mathcal{T}$ where $\Sigma = \mathrm{Sig}(\mathcal{T}' \cup \{A\})$. It suffices to show that

$$\mathcal{T}' \cup \mathcal{T} \models C \sqsubseteq D \text{ iff } \mathcal{T}' \cup \mathcal{T}^* \models C \sqsubseteq D$$

where C and D are $\mathcal{ALC}$-concept descriptions over Σ.

To prove the right to left direction i.e., if $\mathcal{T}' \cup \mathcal{T}^* \models C \sqsubseteq D$ then $\mathcal{T}' \cup \mathcal{T} \models C \sqsubseteq D$. It suffices to show that for any $\mathcal{ALC}$-concept description C, if C is satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}$ then it is satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}^*$ as well. But that is trivial by the choice of $\mathcal{T}^*$ (as in Lemma III.4). Since $\mathcal{T} \models \mathcal{T}^*$, therefore any model of $\mathcal{T}$ is also a model of $\mathcal{T}^*$. Hence C is also satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}^*$.

Similarly to prove the left to right direction we show that any $\mathcal{ALC}$-concept description C satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}^*$ is also satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}$. Depending on the form of $\mathcal{T}^* \in \Pi_A$ we make the following case distinctions.

1. $\mathcal{T}^* = \{\top \sqsubseteq \bot\}$:
   The case is trivial as no concept description is satisfiable with respect to $\mathcal{T}' \cup \{\top \sqsubseteq \bot\}$.

2. $\mathcal{T}^* = \{\top \sqsubseteq A\}$
   Let $\mathcal{I}$ be a model of $\mathcal{T}'$, $\mathcal{T}^*$ and C, and $\mathcal{J}$ be a model of $\mathcal{T}$. Model $\mathcal{J}$ surely exists because we can assume that TBox $\mathcal{T}$ is consistent. In case it is not, nothing has to be proved then. Lifting both $\mathcal{I}$ and $\mathcal{J}$ to ω we get $\mathcal{I}_\omega$ and $\mathcal{J}_\omega$ respectively. It follows from Lemma II.2 that $\mathcal{I}_\omega$ is a model of $\mathcal{T}'$, $\mathcal{T}^*$, C and $\mathcal{J}_\omega$ is model of $\mathcal{T}$. Since $|\Delta^{\mathcal{I}_\omega}| = |\Delta^{\mathcal{J}_\omega}|$, we can define a bijection $\pi : \Delta^{\mathcal{I}_\omega} \to \Delta^{\mathcal{J}_\omega}$.
   Now we define an interpretation $\mathcal{K}$ as follows:
   - $\Delta^{\mathcal{K}} := \Delta^{\mathcal{J}_\omega}$
   - For each concept name B and role name r in $\mathrm{Sig}(\mathcal{T})$:

   $$B^{\mathcal{K}} := \{d | d \in B^{\mathcal{J}_\omega}\}$$
   $$r^{\mathcal{K}} := \{(d, d') | (d, d') \in r^{\mathcal{J}_\omega}\}$$

   and if B and r in $\mathrm{Sig}(\mathcal{T}' \cup \{A\})$ then

   $$B^{\mathcal{K}} := \{\pi(d) | d \in B^{\mathcal{I}_\omega}\}$$
   $$r^{\mathcal{K}} := \{\langle \pi(d), \pi(d') \rangle | (d, d') \in r^{\mathcal{I}_\omega}\}$$

   Here $\mathcal{K}$ is well-defined because $\mathrm{Sig}(\mathcal{T}') \cap \mathrm{Sig}(\mathcal{T}) \subseteq \{A\}$ and $|A^{\mathcal{I}_\omega}| = |A^{\mathcal{J}_\omega}| = |\Delta^{\mathcal{K}}|$ hence $A^{\mathcal{K}} = \Delta^{\mathcal{K}}$. It is easy to show that $\mathcal{K}$ is a model of $\mathcal{T}$ and of $\mathcal{T}'$, $\mathcal{T}^*$ and C. Thus C is also satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}$.

3. $\mathcal{T}^* = \{A \sqsubseteq \bot\}$
   This case is analogous to the case $\mathcal{T}^* = \{\top \sqsubseteq A\}$ except that A is interpreted by $\emptyset$.

4. $\mathcal{T}^* = \{\neg(\top \sqsubseteq A), \neg(A \sqsubseteq \bot)\}$
   Let $\mathcal{I}$ be a model of $\mathcal{T}' \cup \mathcal{T}$ and C and $\mathcal{J}$ a model of $\mathcal{T}$ such that $A^{\mathcal{J}} \neq \top$ and $A^{\mathcal{J}} \neq \bot$. The existence of such a model $\mathcal{J}$ is guaranteed as $\mathcal{T} \models \mathcal{T}^*$. Now again we lift both the models to ω obtaining $\mathcal{I}_\omega$ and $\mathcal{J}_\omega$ respectively. Again we define a bijection $\pi : \Delta^{\mathcal{I}_\omega} \to \Delta^{\mathcal{J}_\omega}$ which satisfies the following:
   For any $d \in \Delta^{\mathcal{I}_\omega}$,
   - if $d \in A^{\mathcal{I}_\omega}$, then $\pi(d) = d' \in \Delta^{\mathcal{J}_\omega}$ such that $d' \in A^{\mathcal{J}_\omega}$.
   - if $d \in (\neg A)^{\mathcal{I}_\omega}$, then $\pi(d) = d' \in \Delta^{\mathcal{J}_\omega}$ such that $d' \in (\neg A)^{\mathcal{J}_\omega}$.

   Note that we can define such a bijection as $|A^{\mathcal{I}_\omega}| = |A^{\mathcal{J}_\omega}|$ and $|\neg A^{\mathcal{I}_\omega}| = |\neg A^{\mathcal{J}_\omega}|$. Now we define an interpretation $\mathcal{K}$ as follows:

- $\Delta^{\mathcal{K}} := \Delta^{\mathcal{J}_\omega}$
- For any concept name B and role name r if B and r are in $\mathrm{Sig}(\mathcal{T})$ then

$$B^{\mathcal{K}} := \{d | d \in B^{\mathcal{J}_\omega}\}$$
$$r^{\mathcal{K}} := \{(d, d') | (d, d') \in r^{\mathcal{J}_\omega}\}$$

and if B and r in $\mathrm{Sig}(\mathcal{T}' \cup \{A\})$ then

$$B^{\mathcal{K}} := \{\pi(d) | d \in B^{\mathcal{I}_\omega}\}$$
$$r^{\mathcal{K}} := \{\langle \pi(d), \pi(d') \rangle | (d, d') \in r^{\mathcal{I}_\omega}\}$$

Note that for any $d \in \Delta^{\mathcal{I}_\omega}$, π guarantees that if $d \in A^{\mathcal{I}_\omega}$(respectively $d \in (\neg A)^{\mathcal{I}_\omega}$) then $\pi(d) \in A^{\mathcal{J}_\omega}$(respectively $\pi(d) \in (\neg A)^{\mathcal{J}_\omega}$). Hence $\mathcal{K}$ is well-defined as A is the only common symbol between $\mathcal{T}'$, $\mathcal{T}^*$, C and $\mathcal{T}$. Further, $\mathcal{K}$ is a model of $\mathcal{T}' \cup \mathcal{T}$ and C. Hence C is satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}$.

5. $\mathcal{T}^* = \{\neg(A \sqsubseteq \bot)\}$

As $\mathcal{T} \models \mathcal{T}^*$, all we know about the way A is interpreted in a model of $\mathcal{T}$ is that A cannot be interpreted by the empty set. Thus we have the following sub-cases:

- A is interpreted by the whole domain i.e., $\top \sqsubseteq A$, therefore the case is analogous to Case 2.
- A is interpreted by some none empty set but not by $\top$ i.e., $\neg(A \sqsubseteq \bot)$ and $\neg(\top \sqsubseteq A)$ and therefore is analogous to case 4.

6. $\mathcal{T}^* = \{\neg(\top \sqsubseteq A)\}$

Similar to case 5 except that here we have the following sub-cases:

- A is interpreted by the empty set i.e., $A \sqsubseteq \bot$ hence can be treated like case 3.
- A is interpreted by some none empty set other than the domain i.e., $\neg(A \sqsubseteq \bot)$ and $\neg(\top \sqsubseteq A)$ and therefore again is analogous to case 4.

7. $\mathcal{T}^* = \emptyset$

This is the most interesting case in the sense that we don't know the way concept name A is interpreted in a model of TBox $\mathcal{T}$. Hence we have to look for all the possibilities that concept name A is interpreted in a model of $\mathcal{T}$. Let $\mathcal{I}$ be a model of $\mathcal{T}'$ and C. Then we have the following sub-cases:

(a) $\mathcal{I} \models \top \sqsubseteq A$

This means that the concept name A is interpreted by the whole domain. Hence is analogous to Case 2.

(b) $\mathcal{I} \models A \sqsubseteq \bot$

This means that the concept name A is interpreted by the empty set. Hence is analogous to Case 3.

(c) Neither (a) nor (b)

Let $\mathcal{J}_1$ be models of $\mathcal{T}$ and A and $\mathcal{J}_2$ of $\mathcal{T}$ and $\neg A$. Models $\mathcal{J}_1$ and $\mathcal{J}_2$ surely exist because $\mathcal{T} \models \mathcal{T}^*$ and its not the case that $\mathcal{T}^* = \{\top \sqsubseteq A\}$ or $\mathcal{T}^* = \{A \sqsubseteq \bot\}$. Let $\mathcal{J}$ be the disjoint union of $\mathcal{J}_1$ and $\mathcal{J}_2$. Then $\mathcal{J}$ is a model of $\mathcal{T}$, A and $\neg A$. By lifting both $\mathcal{I}$ and $\mathcal{J}$ to ω, we get $\mathcal{I}_\omega$, and $\mathcal{J}_\omega$. Lemma II.2 ensures that $\mathcal{I}_\omega$ is a model of $\mathcal{T}'$ and C, and $\mathcal{J}_\omega$ a

model of $\mathcal{T}$, A and $\neg A$. Now we define a bijection $\pi := \Delta^{\mathcal{I}_\omega} \to \Delta^{\mathcal{J}_\omega}$ such that

- if $d \in A^{\mathcal{I}_\omega}$ then $\pi(d) \in A^{\mathcal{J}_\omega}$
- if $d \in (\neg A)^{\mathcal{I}_\omega}$ then $\pi(d) \in (\neg A)^{\mathcal{J}_\omega}$

Note that $|A^{\mathcal{I}_\omega}| = |A^{\mathcal{J}_\omega}| = |\neg A^{\mathcal{I}_\omega}| = |\neg A^{\mathcal{J}_\omega}|$, therefore, π is well-defined.

We define an interpretation $\mathcal{K}$ as follows:

- $\Delta^{\mathcal{K}} := \Delta^{\mathcal{J}_\omega}$
- For concept name B and role name r if B and r are in $\mathrm{Sig}(\mathcal{T})$ then

$$B^{\mathcal{K}} := \{d | d \in B^{\mathcal{J}_\omega}\}$$
$$r^{\mathcal{K}} := \{(d, d') | (d, d') \in r^{\mathcal{J}_\omega}\}$$

and if B and r in $\mathrm{Sig}(\mathcal{T}' \cup \{A\})$ then

$$B^{\mathcal{K}} := \{\pi(d) | d \in B^{\mathcal{I}_\omega}\}$$
$$r^{\mathcal{K}} := \{\langle \pi(d), \pi(d') \rangle | (d, d') \in r^{\mathcal{I}_\omega}\}$$

Again $\mathcal{K}$ is well-defined because $\mathrm{Sig}(\mathcal{T}') \cap \mathrm{Sig}(\mathcal{T}) \subseteq \{A\}$. Hence $\mathcal{K}$ is a model of $\mathcal{T}$. Similarly $\mathcal{K}$ is also a model of $\mathcal{T}'$, and C. Thus, C is also satisfiable with respect to $\mathcal{T}' \cup \mathcal{T}$.

$\blacksquare$

Theorem III.3 shows that when importing a single concept name, say A, in an ontology from some other ontology, one of the TBoxes in Π_A(as in Theorem III.3) serves as a module. The important thing here is the relaxation of the condition for a module being a subset of the original ontology. In practical problems when we are only interested in the conclusion of a TBox only, our notion of signature substitute reflects a black box behavior of a module(of course, in the case of import of a single concept name).

IV. Practical Issues

Theorem III.3 shows us that when importing a single concept name, we don't get much information in the sense that the set of TBoxes (as Π_A Thoerem III.3) always remains the same. In literature we come accross algorithms for extracting module. In case of single concept name import, by using such algorithms we may end up with module which is not the minimal i.e.,the extracted module may contain many other concept names and role names as well. In contrast, we have shown that for importing a concept name A, one of the TBoxes in Π_A(as in Theorem III.3) serves as the required module. Nevertheless, this does not mean that such algorithms are not of much practical interest. We don't get much from a single concept name import when we are interested in consequences of an ontology. But there are applications (e.g., modeling) where the existing notions and algorithms are applicable and useful.

The complexity of the task of determining the TBox $\mathcal{T}^*$ in Π_A (as in Theorem III.3), which fulfills the requirement of a module, depends on the underlying Description Logic in which the TBox $\mathcal{T}$ (as in Theorem III.3) is formulated. In general we can perform the task as follows:

- For all $\mathcal{T}^* \in \Pi$, check whether $\mathcal{T} \models \mathcal{T}^*$. This can be done by checking whether $\mathcal{T} \models C \sqsubseteq D$ for all

$\{C \sqsubseteq D\} \subseteq \mathcal{T}^*$. But there can be at most two GCIs in a $\mathcal{T}^* \in \Pi_A$. Hence this later task is ExpTime-Complete in $\mathcal{ALC}$ for example. Further there are only six $\mathcal{T}^* \in \Pi_A$ to be checked. Finally determining the most specific $\mathcal{T}^*$ (see Lemma III.4) needs a constant time. Overall the task of determining TBox $\mathcal{T}^*$ is ExpTime-Complete in the case of $\mathcal{ALC}$.

V. Conclusion

In this paper, we have studied the import of a single concept name for the purpose of reuse. Making a slight change (i.e. dropping the condition for a module being a subset of the original ontology) in the notion of modularity, we have seen that we end up always with the same set of modules for a single concept name. We have also seen that our notion of modularity reflects the black-box behavior of modules. Finally we have seen that the complexity of computing such a module depends on the underlying ontology language in which the original ontology is formulated.

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