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Some properties of the hypergeometric functions of one variable

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ABSTRACT : In this paper we introduced the novel concept of basic hypergeometric series and the hypergeometric function. We express many of common mathematical function in terms of the hypergeometric function. Gauss' contigous relations, some integral formulas, Recurrence relations, transformation formulas, values at the special points.

Keywords : Transformation formulas, Integral representation, Gauss' contiguous relations, Recurrence relations, Gauss's continued fraction, Special cases.

I. INTRODUCTION

The basic hypergeometric series ${}_2\phi_1\,(q^{\alpha}, q^{\beta}; q^{\gamma}; q, x)$ was first considered by *Eduard Heine* (6). It becomes the hypergeometric series $F(\alpha, \beta; \gamma; x)$ in the limit when the base q is 1. Hypergeometric series were studied by *Euler*, but the first full systematic treatment was given by Gauss [5], Studies in the nineteenth century included those of Ernst [7].

The term ''hypergeometric function'' sometimes refers to the generalized hypergeometric function. The Gaussian or ordinary hypergeometric function ${}_2F_1\,(a, b; c; z)$ is a special function represented by the hypergeometric series, that inclueds many other special functions as special or limiting cases. For systematic lists of some of the many thousands of publishded identities involving the hpergeometric function, are given by Arthur *et al.* [3] Abaramowitz and Stegun [1] and Daalhuis [2].

The generalized basic hypergeometric series cf. Gasper and Rahman [4] is given by

$${}_r\Phi_s\,(a_1, \dots, a_r; b_1, \dots, b_s; q, x) = {}_r\Phi_s \begin{bmatrix} a_1, \dots, a_r\ ; \\ q,\, x \\ b_1, \dots, b_s\ ; \end{bmatrix}$$

$${}_r\Phi_s\ [(a_r);\ (b_s);\ q,\ x] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)n}{(q, b_1, \dots, b_s; q)n}\ x^n \{(-1)^n q^{n(n-1)/2}\}^{(1+s-r)} \qquad \dots(1.1)$$

where for real or complex a

$$(a; q)_n = \begin{cases} 1 & ;\, if\ \ n = 0 \\ (1-a)\,(1-aq)\dots(1-aq^{n-1}) & ;\, if\ \ n \in N, \end{cases} \qquad \dots(1.2)$$

is the q-shifted factorial, r and s are positive integers, and variable x, the numerator parameters $a_1, \dots, a_r$, and the denominator parameters $b_1, \dots, b_s$ beingany complex b_j 6 = q – m, $m = 0, 1, \dots ; j = 1, 2, \dots, s$.

If $|q| < 1$, the series (1.1) converges absolutely for all x if r _ s and for $|x| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|x| < |b_1 b_2 \dots b_s|/|a_1 a_2 \dots a_r|$.

Further, in terms of the q-gamma function, (1.2) can be expressed as

$$\Gamma_q(a) = \frac{(q;\, q)\,\infty}{(q^a;\, q)\,\infty\,(1-q)^{a-1}} = \frac{(q;\, q)_{a-1}}{(1-q)^{a-1}}, \qquad \dots(1.3)$$

where the q-gamma function (cf. Gasper and Rahman [4]) is given by

$$\Gamma_q(a) = \frac{(q;\, q)\,\infty}{(q^a;\, q)\,\infty\,(1-q)^{a-1}} = \frac{(q;\, q)_{a-1}}{(1-q)^{a-1}}, \qquad \dots(1.4)$$

The theory of basic hypergeometric functions of one and more variables has a wide range of applications in various fields of Mathematical, Physical and Engineering Sciences, namely-Numbe theory, Partition theory, Combinatori-alanalysis, Lie theory, Fractional calculus, Integral transforms, Quantum theory *etc*. In the present work, we express the generalized basic hypergeometric function ${}_{s+1}\Phi_s(.)$ in terms of an iterated q-integrals involving the q-Gauss hypergeometric function. Using q-contiguous relations for ${}_2\Phi_1(.)$, we obtain somerecurrence relations for the generalized basic hypergeometric functions of one variable. The above mentioned technique is a q-version of the technique used by Galu'e and Kalla [8].

II. TRANSFORMATION FORMULAS

Euler's transformation is

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b; c; z). \qquad \dots(2.1)$$

It follows by combining the two Pfaff transformations

$${}_2F_1(a, b; c; z) = (1-z)^{-b}\ {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right).$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \qquad (2.2)$$

which in turn follow from Euler's integral representation.

III. INTEGRAL REPRESENTATION

In this section, we express the generalized basic hypergeometric function ${}_r\Phi_s(.)$ (for $r = s + 1$) in terms of an iterated integral involving the basic analogue of Gauss hypergeometric function.

Theorem : Let $Re(bs - i) > 0$, for all $i = 0, 1, \dots, s-2$ and $|q| < 1$, then the iterated q-integral representation of ${}_{s+1}\Phi_s(.)$ is given by

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \dots a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, ..b_s & ; \end{matrix}\right] $$

$$ = \prod_{i=0}^{s-1} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}\ b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \underbrace{\int_0^1 \cdots \int_0^1}_{(s-1)\text{ times}} \prod_{i=0}^{s-2} t_{i+1}^{a_{2+1-s-1}} (t_i + 1q; q)\, b_{2-i-a_2+1-i^{-1}} $$

$$ \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} & ; \\ & q, t_{s-1 \dots t_2 t_1 x} \\ q^{\gamma} & ; \end{matrix}\right] d_q t_{s-1} \dots d_q t_2 d_q t_1 $$

where $|x| < 1$ and $|t_{s-1} \dots t_2 t_1 x| < 1$.

Proof. To prove the theorem, we consider the well-known q-integral representation of r_ s(.), namely

$$ {}_r\Phi_s\left[\begin{matrix} a_1, \dots a_r & ; \\ & q, x \\ b_1, \dots, b_s & ; \end{matrix}\right] = \left[\begin{matrix} b_s \\ a+1, b_{s-}\ a_{s+1} \end{matrix}\right] $$

$$ \times \int_0^1 t^{a_r+1} (tq; q)\, b_{s-a_r-1}\ {}_{r-1}\Phi_{s-1} $$

$$ \left[\begin{matrix} a_1, \dots, a_{r-1}\,; & \\ & q, tx \\ b_1, \dots, b_{s-1}\,; & \end{matrix}\right] d_q t, \qquad \text{...(3.2)} $$

which is the generalization of q-analogue of Euler's integral representation, namely (cf. Gasper and Rahman [4])

$$ {}_2\Phi_1\left[\begin{matrix} q^a, q^b & ; \\ & q, x \\ q^c & ; \end{matrix}\right] $$

$$ = \frac{\Gamma_q(c)}{\Gamma_q(b)\,\Gamma_q(c-b)} \int_0^1 t^{b-1} (tq; q)_{c-b-1} $$

$$ \times {}_1\Phi_0\left[\begin{matrix} q^a & ; \\ & q, tx \\ - & ; \end{matrix}\right] d_q t. \qquad \text{...(3.3)} $$

Therefore, relation (3.2) can also be written as

$$ {}_{s+1}\Phi_s\left[\begin{matrix} a^{\alpha}, q^{\beta}, a_3, a_4, \dots, a_s, a_{s+1} & ; & q, x \\ a^{\gamma}, b_2, b_3, \dots b_s & ; & \end{matrix}\right] $$

$$ = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1,}\ b_s - 1 - a_{s+1} \end{matrix}\right] $$

$$ \times \int_0^1 t^{a_2+1} (t_1 q; q) b_{2-a_2+1-1}\ {}_s\varphi_{s-1} $$

$$ \left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \dots, a_s; & \\ & q, t_{1x} \\ q^{\gamma}, b_2, b_3, \dots, b_{s-1} & ; \end{matrix}\right] d_q t_1. \qquad \text{....(3.4)} $$

Repeating the process in the right-hand side of (3, 4), we get

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \dots, a_s, a_{s+1} & ; & \\ & & q, x \\ q^{\gamma}, b_2, b_3, \dots b_s & ; & \end{matrix}\right] $$

$$ = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1,}\ b_s - a_{s+1} \end{matrix}\right] $$

$$ \times \left[\begin{matrix} b_{s-1} \\ a_s, b_{s-1} - a_s \end{matrix}\right] \int_0^1 \int_0^1 t_2^{a_2-1} (t_2 q; q)_{b_{2-1} - a_{2-1}} $$

$$ t_1^{a_{2+1}-1} (t_1 q; q) b_2 - a_{2+1} - 1 $$

$$ \times {}_{s-1}\Phi_{s-2}\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \dots, a_{s-1} & ; & \\ & & q, t_2 t_1 x \\ q^{\gamma}, b_2, b_3, \dots, b_{s-2} & ; & \end{matrix}\right] d_q t_2 d_q t_1. $$

$$ \text{...(3. 5)} $$

Successive operation $(s - 3)$ times in the right-hand side of (3.5) leads to the desired result (3.1).

IV. GAUSS' CONTIGUOUS RELATIONS

The six functions ${}_2F_1(a \pm 1, b; c; z)$, ${}_2F_1(a, b \pm 1; c; z)$, and ${}_2F_1(a, b; c \pm 1; z)$ are called contiguous to ${}_2F_1(a, b; c; z)$. Gauss showed that ${}_2F_1(a, b; c; z)$ can be written as a linear combination of any two of its contiguous functions, with rational coefficients in terms of a, b, c, and z.

$$ z\frac{dF}{dz} = z\frac{ab}{c} = F(a+, b+, c+) = a(F(a+) - F) $$

$$ = b(F(b+) - F) $$

$$ = (c-1)(F(c-) - F) $$

$$ = \frac{(c-a)\,F(a-) - (a - c + bz)\,F}{1-z} $$

$$ = \frac{(c-b)\,F(b-) + (b - c + az)\,F}{1-z} $$

$$ = z\,\frac{(c-a)(c-b)\,F(c+) + c(a+b-c)\,F}{c(1-z)} $$

In the notation above, $F = {}_2F_1(a, b; c; z)$, $F(a+) = {}_2F_1(a+1\ b; c; z)$ and so on.

Repeatedly applying these relations gives a linear relation between any three functions of the form ${}_2F_1(a + m,\ b + n;\ c + l;\ z)$, where m, n and l are integers.

V. RECURRENCE RELATIONS

In this section, as an application of the integral representation for ${}_{s+1}\Phi_s(.)$, given by (3.1), we shall derive certain recurrence relation for the generalized-basic hypergeometric series. Using the relation between q-contiguous basic-hypergeometric functions [4,p.22].

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma-1} ; \end{matrix} q, x\right] - {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] $$

$$ = q^{\gamma} x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} {}_2\Phi_1 $$

$$ \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q^{\gamma+1} ; \end{matrix} q, x\right] \qquad ...(5.1) $$

we get

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, t_{s-1} \ldots t_2 t_1 x\right] $$

$$ = {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma-1} ; \end{matrix} q, t_{s-1} \ldots t_2 t_1 x\right] $$

$$ - q^{\gamma} t_{s-1} \ldots t_2 t_1 x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} $$

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q^{\gamma+1} ; \end{matrix} q, t_{s-1} \ldots t_2 t_1 x\right] \qquad ...(5.2) $$

On substituting value from relation (5.2) in the right-hand side of the result (3.1), we have

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1} ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s ; \end{matrix} q, x\right] $$

$$ = \prod_{i=0}^{s-2} \Gamma_q \left[\begin{matrix} b_{s-1} \\ a_{s+1-i},\ b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \underbrace{\int_0^1 \ldots \int_0^1}_{(s-1)\text{ times}} \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1}\, q; q)_{b_{s+1-i}, b_{s-i-} a_{s+1-i}-1} $$

$$ \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma-1} ; \end{matrix} q, t_{s-1} \ldots t_2 t_1 x\right] d_q t_{s-1} \ldots d_q t_2 d_q t_1 $$

$$ - q^{\gamma} x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \prod_{i=0}^{s-2} \Gamma_q \left[\begin{matrix} b_{s-1} \\ a_{s+1-i},\ b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \underbrace{\int_0^1 \ldots \int_0^1}_{(s-1)\text{ times}} \prod_{i=0}^{s-2} t_{i+1}^{1+a_{s+1-i}-1} (t_{i+1}\, q; q)_{b_{s-i}-a_{s+1-i}-1} $$

$$ \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q^{\gamma+1} ; \end{matrix} q, t_{s-1} \ldots t_2 t_1 x\right] d_q t_{s-1} \ldots d_q t_2 d_q t_1 $$

...(5. 3)

Again, on making use of the result (3.1), the above result (5. 3) leads to thefollowing recurrence relation :

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots, a_s, a_{s+1} ; \\ q^{\gamma}, b_2, a_3, \ldots, b_s ; \end{matrix} q, x\right] $$

$$ = {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots, a_s, a_{s+1} ; \\ q^{\gamma-1}, b_2, a_3, \ldots, b_s ; \end{matrix} q, x\right] $$

$$ - q^{\gamma} x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \times \prod_{i=0}^{s-2} \left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] $$

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \ldots, a_s q, a_{s+1} q ; \\ q^{\gamma+1}, b_2 q, a_3 q, \ldots, b_s q ; \end{matrix} q, x\right], \qquad ...(5.4) $$

where $Re(bs - i) > 0$, for all $i = 0, 1, \ldots, s-2$ and $|x| < 1$.

Similarly, if we consider the following q-contiguous relations (cf. Gasper and Rahman [4, p.22])

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] - {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] $$

$$ = q^{\alpha} x \frac{(1-q^{\beta})}{(1-q^{\gamma})} \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q^{\gamma+1} ; \end{matrix} q, x\right] \qquad ...(5.5) $$

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] - {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] $$

$$ = q^{\alpha} x \frac{(1-q^{\gamma-\alpha})(1-q^{\beta})}{(1-q^{\gamma+1})(1-q^{\gamma})} $$

$$ \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q^{\gamma+2} ; \end{matrix} q, x\right], \qquad ...(5.6) $$

and

$$ {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta-1} ; \\ q^{\gamma} ; \end{matrix} q, x\right] - {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q^{\gamma} ; \end{matrix} q, x\right] $$

$$ q^{\alpha} x = \frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})} \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta} ; \\ q^{\gamma+1} ; \end{matrix} q, x\right] \qquad ...(5.7) $$

and make use of the result (3.1), we obtain the following respective recurrence relations for generalized basic hypergeometric functions, namely

$$ {}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1} ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s ; \end{matrix} q, x\right] $$

$$_{s+1}\Phi_s = \left[\begin{matrix} q^{\alpha+1}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s\ ; \end{matrix}\ q, x\right]$$

$$- q^{\alpha} x \,\frac{(1-q^{\beta})}{(1-q^{\gamma})} \times \prod_{i=1}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right]$$

$$_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \ldots a_s q, a_{s+1} q\ ; \\ q^{\gamma+1}, b_2 q, b_3 q, \ldots, b_s q\ ; \end{matrix}\ q, x\right] \quad \ldots(5.8)$$

where $Re(bs - i) > 0$, for all $i = 0, 1, \ldots, s-2$ and $|x| < 1$.

$$_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s\ ; \end{matrix}\ q, x\right]$$

$$= {}_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha+1}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ q^{\gamma+1}, b_2, b_3, \ldots, b_s\ ; \end{matrix}\ q, x\right]$$

$$- q^{\alpha} x \,\frac{(1-q^{\gamma-\alpha})(1-q^{\beta})}{(1-q^{\gamma=1})(1-q^{\gamma})} \times \prod_{i=1}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right]$$

$$_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \ldots a_s q, a_{s+1} q\ ; \\ q^{\gamma+2}, b_2 q, b_3 q, \ldots, b_s q\ ; \end{matrix}\ q, x\right], \quad \ldots(5.9)$$

where $Re(bs - i) > 0$, for all $i = 0, 1, \ldots s-2$ and $|x| < 1$.

$$_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s\ ; \end{matrix}\ q, x\right]$$

$$= {}_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha+1}, q^{\beta-1}, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ q^{\gamma}, b_2, b_3, \ldots, b_s\ ; \end{matrix}\ q, x\right]$$

$$- q^{\alpha} x \,\frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})} \times \prod_{i=1}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right]$$

$$_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha+1}, q^{\beta}, a_3 q, a_4 q, \ldots a_s q, a_{s+1} q\ ; \\ q^{\gamma+1}, b_2 q, b_3 q, \ldots, b_s q\ ; \end{matrix}\ q, x\right] \quad \ldots(5.10)$$

where $Re(bs - i) > 0$, for all $i = 0, 1, \ldots s-2$ and $|x| < 1$.

VI. GAUSS'S CONTINUED FRACTION

Gauss used the contiguous relations to give several ways to write a quotient of two hypergeometric functions as a continued fraction, for examples

$$\frac{_2F_1(a+1, b; c=1; z)}{_2F_1(a, b; c; z)}$$

$$= \cfrac{1}{1 + \cfrac{\frac{(a-c)b}{c(c+1)} z}{1 + \cfrac{\frac{(b-c-1)(a+1)}{(c+1)(c+2)} z}{1 + \cfrac{\frac{(a-c-1)(b+1)}{(c+2)(c+3)} z}{1 + \cfrac{\frac{(b-c-2)(a+2)}{(c+3)(c+4)} z}{1 + \ddots}}}}} \quad \ldots(6.1)$$

VII. SPECIAL CASES AND VALUES AT SPECIAL POINTS *z*

$$\lim_{q \to 1-} = \Gamma_q(a) = \Gamma(a) \text{ and } \lim_{q \to 1-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n, \quad \ldots(7.1)$$

where $\quad (a)_n = a(a+1) \ldots (a+n-1), \quad \ldots(7.2)$

one can note that the result (2.1) is the q-extension of the known result due to Galu'e and Kalla [3], namely

$$_{s+1}F_s \left[\begin{matrix} \alpha, \beta, a_3, a_4, \ldots a_s, a_{s+1}\ ; \\ \gamma, b_2, b_3, \ldots b_s\ ; \end{matrix}\ x\right]$$

$$= \prod_{i=0}^{s-2} \Gamma\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i} \end{matrix}\right]$$

$$\times \underbrace{\int_0^1 \ldots \int_0^1}_{(s-1)\text{ times}} \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (1-t_{i+1})^{b_{s-i} - a_{s+1-i} - 1}$$

$$\times\ {}_2F_1\left[\begin{matrix} \alpha, \beta\ ; & t_{s-1 \ldots t_2 t_1 x} \\ \gamma\ ; & \end{matrix}\right] dt_{s-1} \ldots dt_2\, dt_1, \quad \ldots(7.3)$$

where $Re(bs - i) > 0$, for all $i = 0, 1, \ldots, s-2$, $|x| < 1$ and $|ts - 1 \ldots t_2 t_1 x| < 1$.

Special values at *z* = 1

Gauss's theorem, named for Carl Friedrich Gauss, is the identity

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad \ldots(7.4)$$

which follows from Euler's integral formula by putting $z = 1$.

Kummer's theorem (z = – 1)

There are many cases where hypergeometric functions can be evaluated at $z = -1$ by using a quadratic transformation to change $z = -1$ to $z = 1$ and then using Gauss's theorem to evaluate the result. A typical example is Kummer's theorem, named for Enst kummer:

$$_2F_1(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)} \quad \ldots(7.5)$$

which follows from Kummer's quadratic transformations

$$_2F_1(a, b; 1 + a - b; -z)$$

$$= (1-z)^{-a}\,{}_2F_1\left(\frac{a}{2}, \frac{1+a}{2} - b; 1 + a - b; \frac{4z}{(1-z)^2}\right)$$

$$= (1-z)^{-a}\,{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; 1 + a - b; \frac{4z}{(1-z)^2}\right) \quad ...(7.6)$$

and Gausss theorem by putting $z = -1$in the first identity.

Values at $z = 1/2$

Gauss's second summation theorem is

$$_2F_1\left(a, b; \frac{1}{2}(1 + a + b); \frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(1 + a + b)\right)}{\Gamma\left(\frac{1}{2}(1+a)\,\Gamma\left(\frac{1}{2}(1+b)\right)\right)} \quad ...(7.7)$$

Bailey's theorem is

$$_2F_1\left(a, 1-a; c; \frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}(1+c)\right)}{\Gamma\left(\frac{1}{2}(c+a)\,\Gamma\left(\frac{1}{2}(1+c-a)\right)\right)}. \quad ...(7.8)$$

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