Acta Universitaria
DIRECCIÓN DE APOYO A LA INVESTIGACIÓN Y AL POSGRADO

The theorem existence and uniqueness of the solution of a fractional differential equation

El teorema de existencia y unicidad de la solución de una ecuación diferencial fraccionaria

Juan Martínez Ortiz*, Leticia Adriana Ramírez Hernández*

ABSTRACT

This article aims to demonstrate, using the Picard-Banach theorem, the proof of the theorem-based existence and uniqueness of the solution of an fractional orden differential equation with a fractional Caputo-type derivative.

RESUMEN

El objetivo de este artículo es demostrar, usando el teorema de Picard-Banach, el teorema de existencia y unicidad de la solución de una ecuación diferencial de orden fraccionario con derivada tipo Caputo.

Recibido: 7 de junio de 2012
Aceptado: 1 de abril de 2013

Keywords:
Fractional differential equations; theorem existence and uniqueness; theorem Picard-Banach.

Palabras clave:
Ecuaciones diferenciales fraccionarias; teorema de existencia y unicidad; teorema de Picard-Banach.

INTRODUCTION

Fractional calculus is a branch of mathematics that deals with operators having a non-integer order, that is, fractional derivatives and fractional integrals. Fractional derivatives and integrals are mathematical operators involving differentiation and integration of an arbitrary (non-integer) order such as $d^{\gamma} f(x) / dx^{\gamma}$, where γ can be taken to be non-integer (Hilfer, 2000; Oldham & Spanier, 1974; Podlubny, 1999; Samko, Kilbas & Maritchev, 1993). In nature, many physical phenomena involve an "intrinsic" fractional order description, so fractional calculus (FC) turns out to be necessary in order to explain them. In many applications FC provides more accurate models of physical systems than ordinary calculus does. However, until now, the theoretical study of the existence and uniqueness of solutions in fractional differential equations has gained relevance (Agrawal, Tenreiro-Machado & Sabatier, 2004; Diethelm & Ford, 2002; Ledesma, 2009a, 2009b; Miller & Ross, 1993).

Basic definitions from fractional calculus

In this section we present the necessary basic definitions concerning fractional calculus. The Caputo derivative is defined as in (Oldham & Spanier, 1974)

$$ {}_{0}^{C}D_t^{\gamma} f(t) = \frac{d^{\gamma}}{dt^{\gamma}} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma+1-n}} \, d\tau, \tag{1} $$

with $n-1 < \gamma \leq n \in I\!N = \{1, 2, \ldots\}$, and $\gamma \in I\!R$ is the order of the fractional derivative and $f^{(n)}(\tau) = d^n f(\tau) / d\tau^n$ represents an ordinary (integer) derivatives, while

*Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas. Carretera a la Bufa con Calzada Solidaridad s/n, Zacatecas, Zac., México. 98000. E-mail: jmartinez_ortiz@yahoo.com; leticiaadrianaramirez@hotmail.com

$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, is the gamma function. The Caputo derivative satisfies the following relations

$$ {}_0^C D_t^\gamma [f(t)+g(t)] = {}_0^C D_t^\gamma f(t) + {}_0^C D_t^\gamma g(t), $$

$$ {}_0^C D_t^\gamma c = 0, \qquad \text{where } c \text{ is a constant.} \tag{2} $$

As can be seen in the above-mentioned formulas, in the Caputo formalism the derivative of a constant is zero, and we can properly define the initial conditions for the fractional differential equations which can be handled by analogy to the classical integer case. The formula for the Laplace transform of the Caputo fractional derivative (Samko *et al.*, 1993) has the form

$$ L\left[\frac{d^\gamma f(t)}{dt^\gamma}\right] = s^\gamma F(s) - \sum_{n=0}^{k-1} s^{\gamma-n-1} f^{(n)}(0), \tag{3} $$

where the derivative $f^{(n)}(0)$ is ordinary. The inverse Laplace transform requires the introduction of the Mittag-Leffler function, which is defined as

$$ E_\beta(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\beta m+1)}, \qquad (\beta > 0), \tag{4} $$

where $\Gamma(\cdot)$ is the Gamma function. When $\beta = 1$, from (4), we have

$$ E_1(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} = \sum_{m=0}^{\infty} \frac{t^m}{m!} = e^t. \tag{5} $$

Therefore, the Mittag-Leffler function is the generalization of the exponential function.

Differential equations

Consider the initial value problem

$$ {}_0^C D_t^\gamma y(x) = f(x, y(x)), \quad y(0) = y_0, \tag{1} $$

where $f \in C([0,h]\times[y_0 - k,\ y_0 + k])$, ${}_0^C D_t^\gamma y$ is the Caputo derivative of the function y and $0 < \gamma < 1$. Since f is continuous, expresion (1) is equivalent to Volterra's integral equation

$$ y(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\gamma-1} f(t, y(t))dt, \ 0 \le t \le h, \tag{2} $$

in other words, any solution of (1) is also solution of (2). Let X be a metric space, with metric ρ.

• Definition 1. *The operator $f: X \to X$ from the metric space X onto itself, is said to be contracive if there exists a number q, $0 < q < 1$, such that $\forall x \in X$, and $\forall y \in Y$ it holds*

$$ \rho(f(x), f(y)) \le q\rho(x, y). \tag{1} $$

• Observation. From this condition, it can be shown that if for any $\varepsilon > 0$ there is a $\delta = \varepsilon$, $\forall x \in X, \forall y \in Y$, for which $\rho(x, y) < \delta$, then

$$ \rho(f(x), f(y)) \le q\rho(x, y) < q\delta < \varepsilon, $$

that is, a contractive operator is uniformly continuous, and therefore continuous for each x de X.

• Definition 2. *A point $x \in X$ is called fixed point for the operator $f: X \to X$, $f(x) = x$.*

The proof of the existence and uniqueness of the solution of a differential equation is supported by the Picard-Banach fixed point theorem.

• Therorem 1. *Any contractive operator that maps a metric space onto itself has a unique fixed point.*

Moreover, if $f: X \to X$ is a contractive operator that maps a metric space onto itself, and a is its fixed point: $f(a) = a$; then for any $x_0 \in X$ the iterative sequence

$$ x_0,\ x_1 = f(x_0),\ x_2 = f(x_1), \ldots,\ x_{n+1} = f(x_n), \ldots $$

converge to a, in addition, if f satisfies (1), then the following estimation holds

$$ \rho(x_n, a) \le \frac{q^n}{1-q}\rho(x_0, f(x_0)). $$

Existence and uniqueness of the solution

In this section theorem 1 will be used to show the existence and uniqueness of a solution for a fractional differential equation.

• Definition 1. *A function real-valued f, defined in the domain G, satisfies the Lipschitz condition with respect to the second variable $y \in B$, if there exists $L > 0$, such that*

$$ |f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|, \tag{1} $$

• Theorem 2. *Let $0 < \gamma < 1$, $I = [0,h] \subset I\!R$ and $J = [y_0 - k, y_0 + k] \subset I\!R$. Let $f: I \times J \to I\!R$ be a continuous bounded function, that is, there exists $M > 0$ such that*

$$ |f(x,y)| \le M, \ \forall\ (x,y) \in I \times J. $$

Also assume that f satisfies Lipschitz condition. If $Lk < M$, then there exists a unique $y \in C([0,h^*])$, where

$$ h^* = \min\left\{h, \left(\frac{k\Gamma(\gamma+1)}{M}\right)^{1/\gamma}\right\}, $$

that holds the initial value differential equation problem ((1) in section 3).

Proof. Let $T := \{y \in C([0,h^*]) : \| y - y_0 \| \le k\}$, since $T \subset C(IR)$ and it is a closed set, T is a complete space. In T define the operator F:

$$Fy(x) = y_0 + \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t, y(t))dt. \qquad (2)$$

Let us verify that (2) satisfies the hypothesis of theorem (1).

First, let us proof that $F : X \to X$. Indeed,

$$| Fy - y_0 | = \left| \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t, y(t))dt \right|$$

$$\le \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \, | f(t, y(t)) | \, dt$$

$$\le \frac{M}{\Gamma(\gamma+1)} x^\gamma \le \frac{M}{\Gamma(\gamma+1)} (h^*)^\gamma \le k$$

so

$$\| Fy - y_0 \| \le k.$$

Therefore F maps T onto itself.

Second, let us see that T is contractive.

$$| Fy - Fz | = \left| \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \{ f(t, y(t)) - f(t, z(t)) \} dt \right|$$

$$\le \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \, | f(t, y(t)) - f(t, z(t)) | \, dt$$

$$\le \frac{L}{\Gamma(\gamma+1)} \| y - z \| x^\gamma \le \frac{L(h^*)^\gamma}{\Gamma(\gamma+1)} \| y - z \| \le \frac{Lk}{M} \| y - z \|,$$

so

$$\| Fy - Fz \| \le \frac{Lk}{M} \| y - z \|,$$

since by hypothesis

$$\frac{Lk}{M} < 1.$$

This shows that T is contractive, and thus has a fixed point. Therefore, the initial value differential equation ((1) in section 2) has a unique solution.

REFERENCES

Agrawal, O. P., Tenreiro-Machado, J. A. & Sabatier, I. (2004). *Fractional Derivatives and Their Applications*. Berlin: Springer-Verlag.

Diethelm, K. & Ford, N. (2002). Analysis of fractional diffential equations. *Journal of Mathematical Analysis and Applications, 265*(2), 229-248.

Hilfer, R. (2000). *Applications of Fractional Calculus in Physics*. Singapore: Ed. World Scientific.

Ledesma, C. T. (2009)a. *Soft Mathematical Notes, 68*-70.

Ledesma, C. T. (2009)b. *Existencia y Unicidad de la Solución de Ecuaciones Diferenciales Ordinarias de Orden Fraccionario* (Undergraduate Thesis). Facultad de Ciencias Físicas y Matemáticas, Universidad Nacional de Trujillo: Trujillo, Perú.

Miller, K. S. & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations.* New York: Wiley.

Oldham K. & Spanier, J. (1974). *The Fractional Calculus.* New York: Academic Press.

Podlubny, I. (1999). *Fractional Differential Equations.* San Diego: Academic Press.

Samko, S. G., Kilbas, A. A. & Maritchev, O. I. (1993). *Fractional Integrals and Derivatives, Theory and Applications*. Langhorne, PA: Gordon and Breach Science Publishers.