Acta Universitaria
DIRECCIÓN DE APOYO A LA INVESTIGACIÓN Y AL POSGRADO

Population dynamics with fractional equations (Predator-Prey)

Dinámica de poblaciones con ecuaciones fraccionarias (Depredador-Presa)

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ABSTRACT

The main objective is to analyze the Predator-Prey model from the fractional calculus perspective. Fractional calculus is a tool for modeling phenomena associated with non-locality. The fractional model developed in Rivero, Trujillo, Vázquez & Velasco (2011), which considers abnormal situations in population growth, will be used.

RESUMEN

El objetivo principal es analizar el modelo Depredador-Presa desde la perspectiva del cálculo fraccionario. El cálculo fraccionario es una herramienta para la modelación de fenómenos asociados a la no localidad. Abordaremos el modelo fraccionario propuesto en Rivero, Trujillo, Vázquez & Velasco (2011), el cual considera situaciones anómalas en el crecimiento de las poblaciones.

INTRODUCTION

Fractional calculus is almost as old as the differential calculus. The concept first appeared in a letter that L'Hôpital sent to Leibniz in 1695, in which he wondered what would happen if the order n of derivative $d^y y/dx^n$, was not an integer number. Leibniz apparently concluded that it was a paradox. Later issues emerged in the non-integer order of the derivative. Authors such as Euler, Laplace, Fourier, and Abel Lacroix approached this subject; the latter was the first one to provide a high-impact application of fractional calculus.

In this paper we address the Predator-Prey dynamics, analyzing the classical model developed by Volterra (Devaney, 2004), and the fractional model set by (Rivero, Trujillo, Vázquez & Velasco, 2011). The usefulness of introducing the fractional-order model lies in the fact that the fractional order controls the speed at which the solution to equilibrium is reached.

METHOD

Classical Model

After his studies of the statistics of fishery in the Adriatic Sea, Volterra (Devaney, 2004) realized that, during World War II for some time afterwards, fishing suffered a considerable decrease, while the predator population increased. This phenomenon motivated him to build a general model to describe the predator-prey population's interaction. The well-known system is as follows:

$$\frac{dx}{dt} = ax - bxy,$$

Recibido: 5 de diciembre de 2012
Aceptado: 10 de abril de 2013

Keywords:
Fractional equations; fractional order; dynamic; populations.

Palabras clave:
Ecuaciones fraccionarias; orden fraccionario; dinámica; poblaciones.

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$$\frac{dy}{dt} = -cy + exy,$$

x = number of elements in prey population,

y = number of elements in predator population,

a = rate of growth of prey population,

d = rate of decrease of predator population,

c = rate of growth of predator population,

b = rate of decrease of prey population.

Let us assume, for the sake of simplicity, that the change in the populations is proportional to the elements in each one, and let $a = b = c = d = 1$. In this case, the trajectories are closed curves. There are two equilibrium points: one at (0, 0), which is a saddle; and $(d/c, a/b)$=(1, 1), a stable center.

RESULTS

Fractional Model

When the fractional derivative is introduced, our classical model takes the form:

$${}^{c}D^{\alpha}_{0+}x = x(1-y),$$

$${}^{c}D^{\beta}_{0+}y = y(-1+x),$$

where $\beta = r\alpha$, $\alpha \in \mathrm{N}$ and ${}^{c}D^{\alpha}_{0+}x$ and ${}^{c}D^{\beta}_{0+}y$ express the fractional order Caputo derivatives. Then, again, the equilibrium points (0, 0), and (1, 1), can be studied.

Near both equilibrium points, using Diethelm algorithm, Rivero *et al.* (2011), developed in the following behavior (figure1).

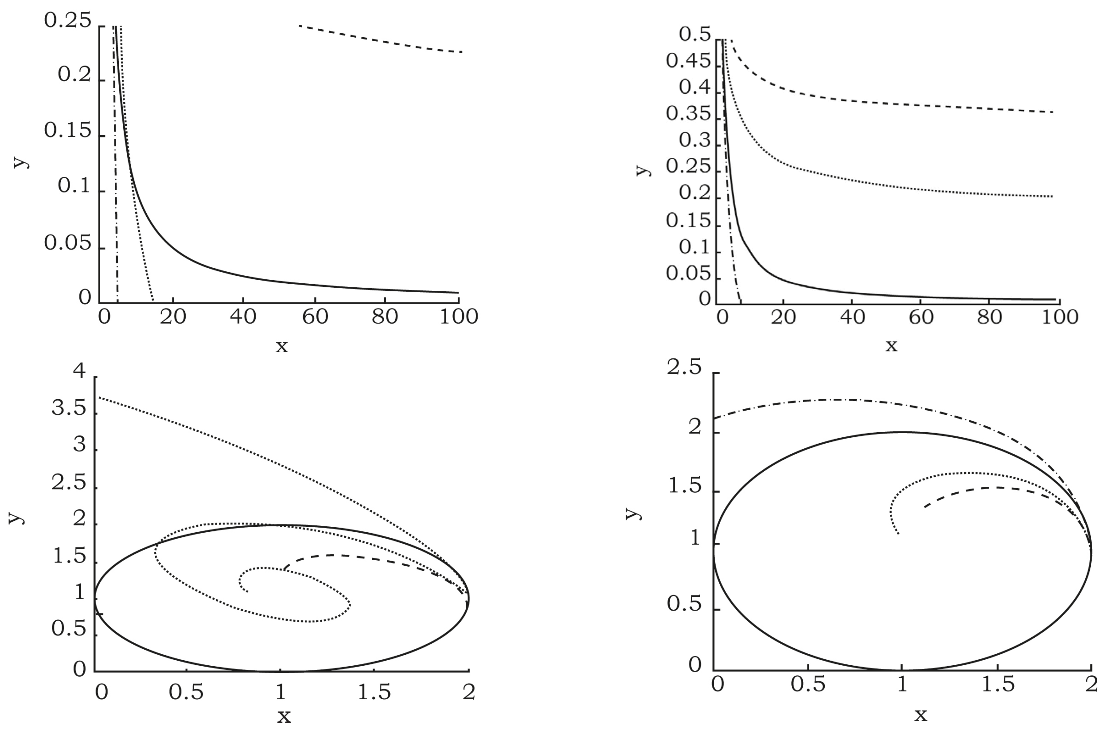

Figure 1. The upper pictures correspond to equilibrium (0, 0). The lower pictures are representative of equilibrium (1, 1). The solid line corresponds to the classical model. The left picture contains the cases α=0.3 and β=0.6 (- -), α=0.6 and β=1.2 (··), α =1.2 and β=2.4 (-·-). The right one contains α=0.3=β (- -), α=0.6=β (··) and α=β=1.2 (-·-).
Source: Rivero *et al.*, (2011).

CONCLUSION

For the first equilibrium point, in both approaches (classical and fractional) the stability of the point remains. In both cases there is a stable curve (Y-direction), and an unestable curve (X-direction).

In addition we observe:

- If $\alpha \leq \beta < 1$ the fractional solutions are slower than those of the classical.

- If $\alpha < \beta < 1$ o $1 < \beta$ then the fractional model solutions are faster than those of the classical.

As for the other equilibrium point (1.1), unlike the centers that are obtained in the classical case, spirals are obtained in the fractional case. In addition:

- If $\alpha \leq \beta < 1$, coils fall within the classical solution.

- If $\alpha < \beta < 1$, spirals have a part inside and one outside of the classical solution.

- If $1 < \alpha \leq \beta$, then the spirals are outside of the classical solution.

Thus, we can conclude that by introducing the fractional approach, parameters α and β allow us to to control the speed of the motion of solutions near equilibrium points.

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