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DESIGN AND ANALYSIS OF A QUADRATIC OPTIMAL CONTROL SYSTEM FOR A TYPE ONE PLANT MODEL

MD. ARIFUR RAHMAN[1] & SYED MUZTUZA ALI[2]

[1]Department of Electronics and Communication Engineering, National University of Singapore, Singapore

[2]School of Mechanical & Aerospace Engineering, Nanyang Technological University, Singapore

ABSTRACT

Automatic control system has played a crucial role in the development of engineering and science and it has become an important and integral part of modern manufacturing and industrial processes. To get from a concrete controlled physical system there are fewsteps to be followed. First, a mathematical model of the physical system is made depending on the existing knowledge of classical physics and this mathematical model can take many forms. The second step in a control system design problem is to decide which desirable properties we want the physical system to satisfy. Very often, these properties can be formulated mathematically by requiring the mathematical model to have certain qualitative or quantitative mathematical properties. Together, these properties form the design specifications. The third, very crucial, step is to design, on the basis of the mathematical model of the physical system, and the list of design specifications, a mathematical model of the physical controller device. The problem of getting from a model and a list of design specifications to a model of a controller is called a control synthesis problem. In this paper, a state-space model has been derived from a type one transfer function plant model. Then the plant's dynamic characteristics have been analyzed and simulated. After that a state feedback controller using the LQR method and finally a rigorous simulation analysis has been done with the designed LQR system with non-zero initial state with zero external inputs.

KEYWORDS: State-Space, Controllability, Observability, Pole-Zero Map, LQR

INTRODUCTION

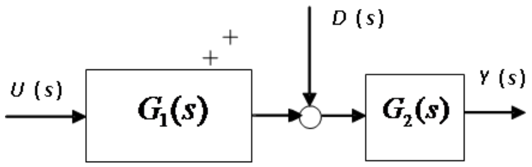
Control system is a useful branch of the engineering education and research and people of control engineering are related with the understanding and controlling segments of their environment, which is often called system. This control system provides useful economic products for society. There are many applications of control engineering such as traffic control systems, chemical processes, hard disk drive system, and robotic systems, and the current challenge of the present control engineers is the modeling and control of these complex systems. Control engineering is composed of the foundations of linear system, Signal processing, circuits and systems and it further integrates this engineering concepts with the basic modeling of mechanical engineering field and applications of control engineering are spreaded to many engineering fields. Therefore control engineering is not limited to any engineering discipline but is important and applicable to all sorts of engineering filed like aeronautical, chemical, mechanical, environmental, civil, and electrical engineering.

The linear quadratic regulator problem, which is commonly abbreviated as LQR, plays an important role in many control design methods (Wilson 1996; Zadeh 1963; Ogata 2002). Quadratic optimal control is not only a powerful design method, but also in many respects it is the fundamental of many systematic control design procedures for linear multiple-input, multiple output (MIMO)systems (Leung 1993; Douglas 1991; Jones 1979. In contrast with the pole-placement method, the theory of optimal control is concerned with operating a dynamic system at minimum cost. If the system dynamics are described by a set of linear differential equations (Datko 1993) which is known as state-space representation and the cost is described by a quadratic functional then it is called the LQR problem. LQR gives infinite gain margin and sufficient Phase margin which is more than 60 degree; therefore LQR ensures the system stability better than any other control design The LQR is an important part of the solution to the LQG problem which is the classical state-space design for disturbance and noise rejection.

Plant Model

A block diagram of the transfer function model of the plant has been given as follows

$$G_1(s) = \frac{1}{s^2 + 2s + 3a}, \qquad G_2(s) = \frac{2}{s},$$



Here insert the value of ‘a’ in the state space representation, according to the given data:

a = parameter of interest = 3.8355

STATE-SPACE MODEL AND ANALYSIS OF DYNAMIC CHARACTERISTICS OF THE PLANT

Sate-Space Model

As we will do state-feedback control system design, it is very important to get the state space model of the given plant. From the state-space model of the system, it is very easy to identify the dynamic behavior of the plant and whether the system is observable and controllable by forming and checking the controllability and observability matrices. From the given open loop plant system, first we find the state variables needed to define the states and then the complete state space of the system. So the given system is expressed in Laplace domain first and it has been converted into time domain and therefore state-space model has been defined by rearranging. The given transfer function block diagram model of the plant as:

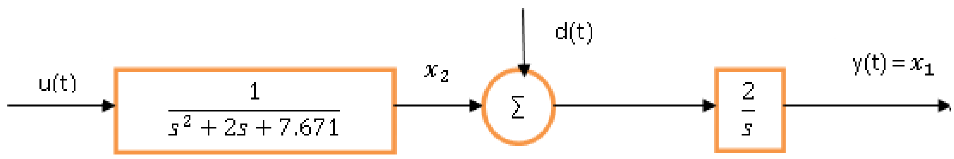


Figure 1: Block Diagram of the Plant

Y(s)=$\frac{2}{s}$[$D(s)$+ $\frac{1}{s^2+2s+7.671}U(s)$]

Now defining as Y(s)= X_1(s) and $\frac{1}{s^2+2s+7.671}$ U(s)= X_2(s)

So now we have, X_1(s) = $\frac{2}{s}[D(s)+X_2(s)]$

Therefore, $\dot{x_1}(t)=2[x_2(t)+d(t)]$

$\dot{x_1}(t)=0.\ x_1(t)+2x_2(t)+\ 0.\ x_1(t)+\ 0.u(t)+1.\ d(t)$ (1)

Where $d(t)$ is disturbance signal.

Also we can see that another state variable

$X_2(s) = \frac{1}{(s^2+2as+3a)} U(s)$

- $\ddot{x_2}(t)+2\,\dot{x_2}(t)+7.671x_2(t)=u(t)$

Now let $x_2(\dot{t}) = x_3(t)$

- $x_3(\dot{t}) + 2\dot{x_2}(t)+7.671x_2(t)=u(t)$
- $x_3(\dot{t}) = -2\dot{x_2}(t)-7.671x_2(t)+u(t)$

So we have

$\dot{x_2}(t)=0.x_2(t) + 1.x_3(t)$

- $\dot{x_2}(t)=0.\,x_1(t)+0.x_2(t) + 1.x_3(t)+ 0.u(t)+0.\,d(t)$ (2)

And $x_3(\dot{t}) =-7.671x_2(t) -2x_3(t)+u(t)$

- $x_3(\dot{t}) = 0.x_1(t)-7.671x_2(t) -2x_3(t)+1.u(t)+ 0.d(t)$ (3)

From equation (1.1)-(1.3) we can find the complete state space model of the plant as :

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ x_3(\dot{t}) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -7.671 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} d(t) \quad (4)$$

Also output equation y(t) can be written as:

$$y(t)= x_1(t) = [1\ \ 0\ \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

State-space Model, $x= Ax + Bu + B_w\, d$

$y=Cx +Du$

where $A= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -7.671 & -2 \end{bmatrix}$, $B= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $B_w = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $C= [1\ \ 0\ \ 0]$ and $D= 0$

Dynamic Characteristics of the Plant

For analysis of dynamic characteristics of the plant, we here consider the transfer function between the output and the input u (t) by ignoring the disturbance signal

So here, open-loop transfer function model

$G(s) = \frac{2}{s(s^2+2s+7.671)} U(s)$

And the state-space model (by ignoring the disturbance input)

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ x_3(\dot{t}) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -7.671 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = x_1(t) = [1\ \ 0\ \ 0]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Pole-Zero Map

Here for this plant has one pole at the origin and two complex poles at -1 ± 2.3814i. So the system is marginally stable.

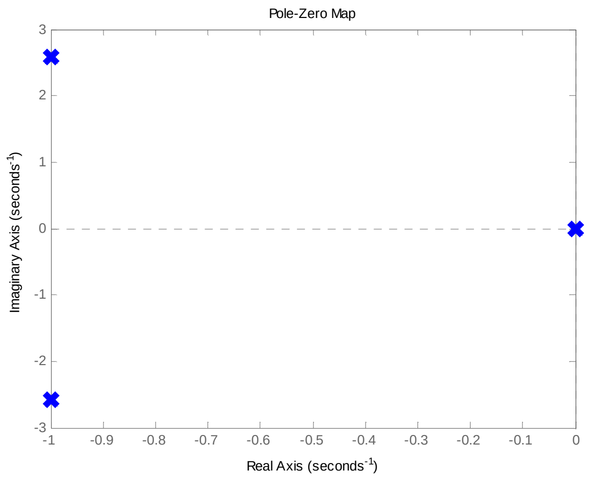


Figure 2: Pole-Zero Map of the Plant

Unit Step Response of the Plant

By observing the step response of the system we can check the Bounded Input Bounded Output (BIBO) stability criteria for the system.

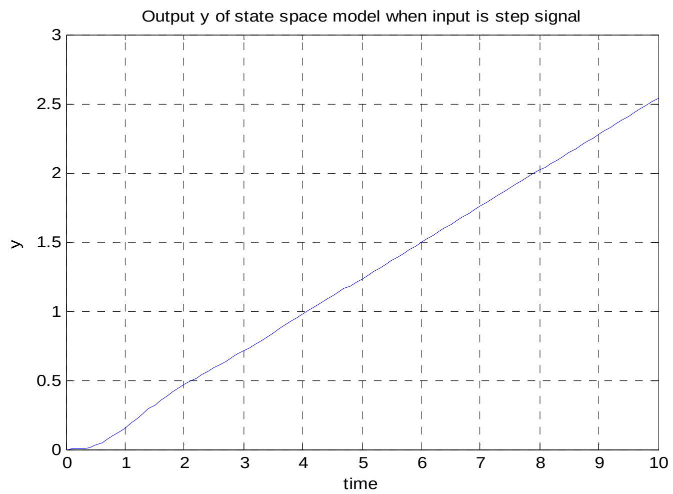


Figure 3: Unit Step Response of the Plant

Responses of the Individual States of the Plant

By considering the initial states of the system as zero if we plot the states of the system then we can get the Figure 4. From this figure it can be seen that, state variable x_1 is showing the same response as y as output $y=x_1$ and other two states become stable after some time.

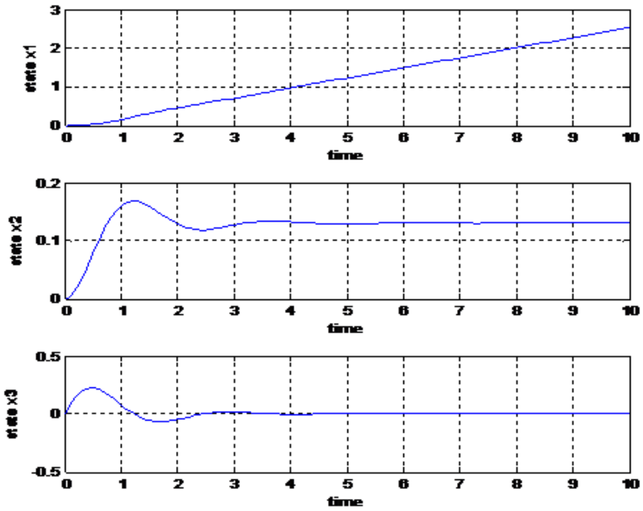


Figure 4: Response of Each State of the Plant

Controllability

The controllability matrix of the given state space system can be found out from the following formula

W_c= [B ABA^2B ………………….A^{n-1}B].

We know that according to controllability theorem, for a system to be controllable the controllability matrix W_c has to be non-singular or full row rank.

For our plant we have n=3, so we have n-1 =2, so the controllability matrix is:

W_c= [B ABA^{2}B]

The controllability matrix obtained using matlab code is:

$$W_c = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -2 & -3.671 \end{bmatrix}$$

The rank of this matrix was found to be 3. The full row rank and we can also check the determinant of this matrix

Det (W_c) = -2, which is non-zero. SoW_c is non-singular. Therefore we can conclude that the Plant is controllable.

Observability

The observability matrix of the given state space system can be found out from the following formula

$W_O = [CCACA^2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots . CA^{n-1}]^T$

We know that according to observability theorem, for a system to be controllable the observability matrix W_O has to be nonsingular or full column rank.

For our plant we have n=3, so we have n-1 =2, so the observability matrix is:

$W_O = [CCACA^2 \ ]^T$

The observability matrix obtained using matlab is:

$$W_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The rank of this matrix was found to be 3. The full column rank and we can also check the determinant of this matrix

Det(W_o) = 4, which is non-zero. So W_o is non-singular.Therefore we can conclude that the Plant is observable.

QUADRATIC OPTIMAL CONTROL

State Feedback Controller Using Linear Quadratic Regulator (LQR)

Given a plant specifications to be met by control design, the specifications are first cast into a specific index or cost function, and the control is sought to minimize the cost function. This cost function can be minimized by designing a state feedback controller for various values of the optimal control and cost minimizing parameters like Q and R.

The system is described by the standard linear state space model as:

$\dot{x}$(t) =Ax(t) + Bu(t), x(0)$\neq$ 0,

y(t) = Cx(t)

The objective is to bring the non-zero initial state to zero. The cost function J is scalar and given as:

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt$$

The Weighting matrices Q and R are symmetric and appear most often in the diagonal form. In addition, it is assumed that Q is semi-positive and R is positive. The optimal control has the form of state feedback:

$$u = -Kx$$

Where K is a constant matrix if A,B,C,Q and R are constant.

The Algebraic Riccati Equation (ARE) is given as:

$A^T P + PA + Q = PBR^{-1}B^T P,$ where P is nxn symmetric matrix

Using this Algebraic Ricatti Equation and searching for minimizing the cost function, it can be derived as:

$u=-R^{-1}B^T Px=-Kx$

Here using the above algorithm, we will design a state feedback controller using LQR method, simulate the designed system and show the state responses to non-zero initial state with zero external inputs. The LQR problem basically is the trade of where we give less weightage to the undesirable parameters and high weightage to the desired ones. Usually Q and R are selected to be diagonal so that specific state and control variables are penalized individually with higher weightings if their response is undesirable. In this case we will choose Q and R as follows, and use LQR method to get the feedback gain K, and make simulation under zero input situations.

Q= $\begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{bmatrix}$, and R=1,

As it is a SISO system, we set the initial condition as, $x_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. As MATLAB has the library function named 'lqr' which implements the above LQR algorithm, therefore by using the 'lqr' command in MATLAB we get the feedback gain K.

Reponses of the Plant with State Feedback Controller Using LQR

Using the MATLAB ‘lqr’ function we can extract the gain matrix K, solution of the Algebraic Ricatti Equation (ARE), P and the eigen values of the closed loop system.

Initially we choose as, Q= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and R=1,

Then after simulation we have found the K,P and Eigen values λ . The response of the individual states are shown in Figure 5; output response of the closed loops system is shown in Figure 6 and control signal plot is shown in Figure 7.

K = [1 0.6927 0.5269]

$$P = \begin{bmatrix} 4.1819 & 2.5269 & 1 \\ 2.5269 & 3.7927 & 0.6927 \\ 1 & 0.6927 & 0.5269 \end{bmatrix},$$

$$E = \begin{bmatrix} -0.2571 \\ -1.1349 + 2.548i \\ -1.1349 - 2.548ii \end{bmatrix}$$

As it can be seen clear from this result that eigen values are all have negative real parts, so the closed loop system becomes stable. And from the figures, it can be seen that output and all the states reaches to zero steady state value after some time.

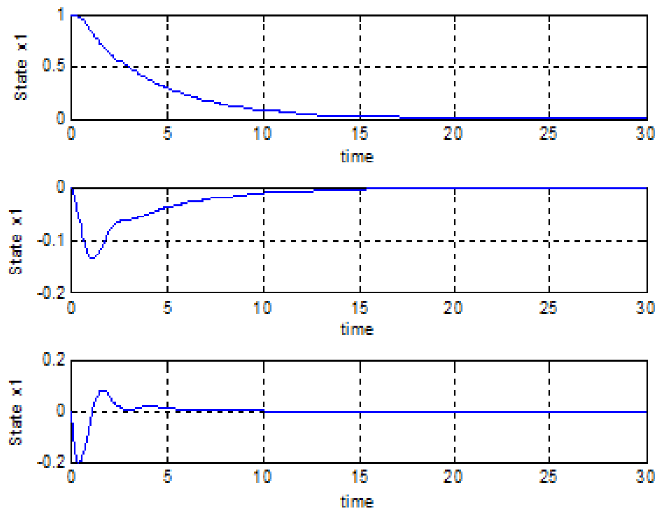
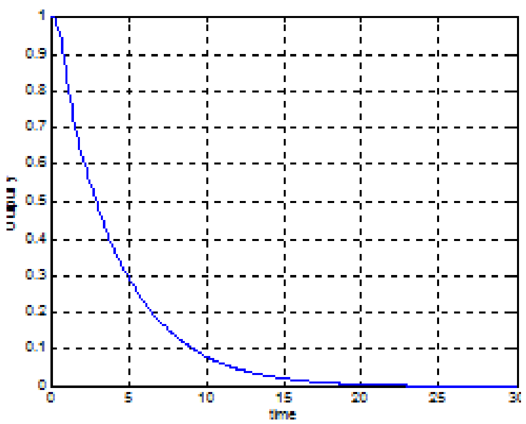


Figure 5: Response of Individual States of the System with LQR



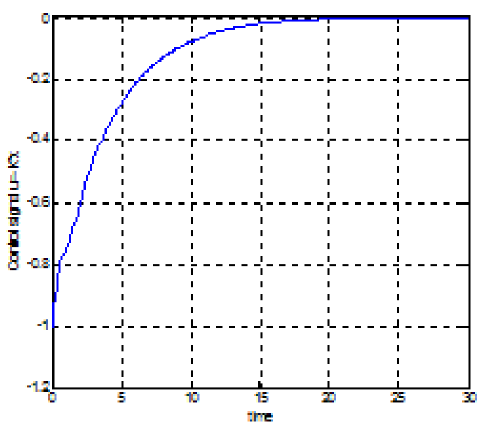


Figure 6: Output Response of the System with LQR **Figure 7: Control Signal of the System with LQR**

Reponses of the Plant for Changing Values of Q of LQR

Here first we define

$$Q = \begin{bmatrix} \boldsymbol{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{q} \end{bmatrix}, \text{ and } R=1$$

Now we change the values of the Q and see the effect on the each state variable now let us choose q as 5,10,30,100 with the value of R=1, and see the plot of all the states variable, output response and control signal for all these values together in a single graph. Figure 8 show the individual state response for different Q values, Figure 9 shows the output response for different Q values and Figure 10 shows the control signal plot for different Q values.

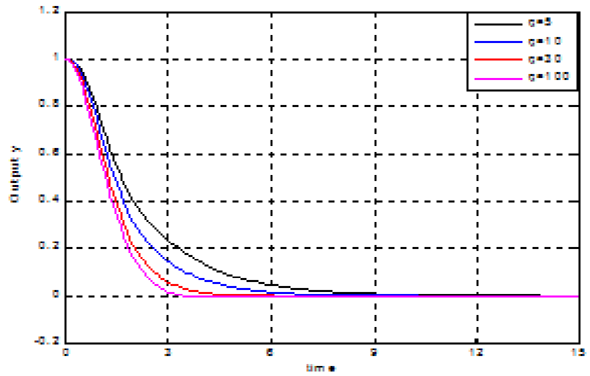


Figure 8: Output Response for Different Q with R=1

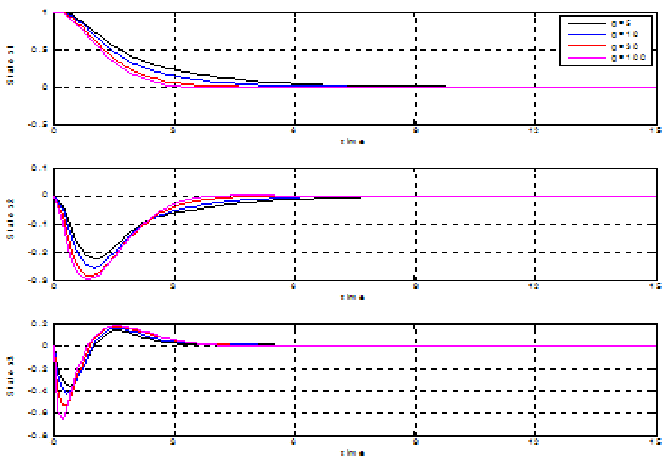


Figure 9: Individual Response of States for Different Q with R=1

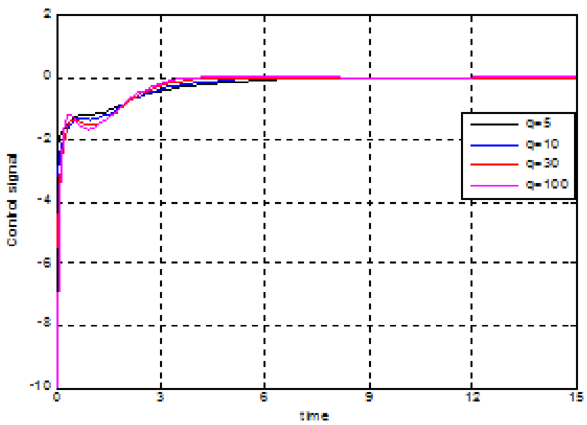


Figure 10: Control Signal for Different Q with R=1

From the above four figures we can analysis the results. From Figure 9 and Figure 8 it can be seen that, as the value of Q increases the all the states and the output reaches the steady state value zero much faster , that's why for $Q = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix}$, the response is the fastest and for $Q = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ the response is the slowest here. This is because we are increasing the weights for the states and making the weight constant for control signal. Therefore whatever the control signal the response is becoming faster. If we look at the control signal plot in Figure 10, we can see that control signal size is little increasing with the increase of the Q since we are not increasing the weight R.

Reponses of the Plant for Changing Values of R of LQR

For the last simulation the value of R was kept constant. Now we change the values of the R with keeping Q constant and see the effect. Now for $Q = \begin{bmatrix} \boldsymbol{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{q} \end{bmatrix}$, let us choose q=1 and the value of R as 1,5,20,50 and see the plot of all the states variable, output response and control signal for all these values together in a single graph. Figure 12 shows the individual state response for different Q values, Figure 11 shows the output response for different Q values and Figure 13 shows the control signal plot for different Q values.

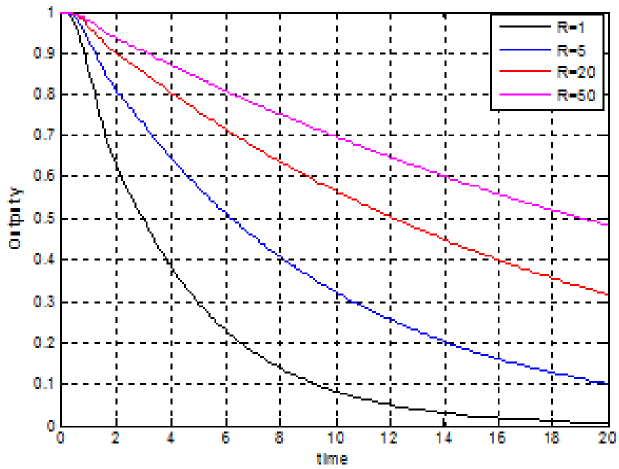


Figure 11: Output Response for Different R and q=1

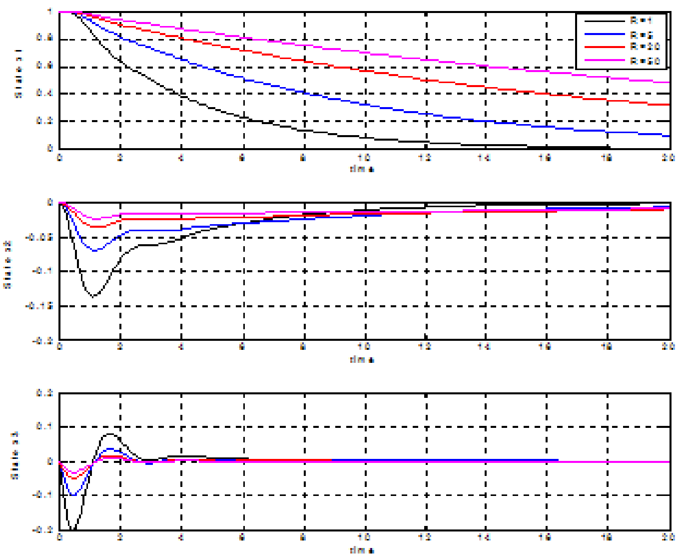


Figure 12: Individual Response of States for Different R and q=1

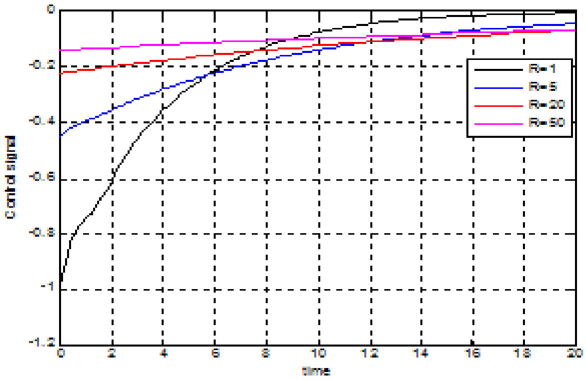


Figure 13: Control Signal for Different R with q=1

From the above four figures we can analysis the results. From Figure 11 and Figure 12 it can be seen that, as the value of R increases the all the states and the output reaches the steady state value zero much slowly, that's why for R=50, the response is the slowest and for R=1, the response is the fastest here. This is because we are increasing the weight for the control signal, so the control effort becomes smaller; therefore it takes more time for the output/states to reach the steady-state. If we look at the control signal plot in Figure 10, we can see that control signal size is decreasing with the increase of the R.

CONCLUSIONS

From the specified plant model the state-space model has been derived and dynamic characteristics of the plant has been studied. It was found that the system is not BIBO stable and the controllability and observability of the system was defined and it was found that the system is both observable and controllable. For the optimal control of the system LQR controller was designed using state feedback and the various results were plotted for various values of Q and R. From the simulation analysis, the effects of the values of Q and R can be easily depicted from the response. The larger the elements of Q are, the larger are the elements of the gain matrix K, and the faster the state variables approach zero, while on the other hand, the larger the elements of R, the smaller the elements of K and the slower the response. So we can conclude that higher values of R make the system sluggish and higher the value of K, the state feedback is higher, the state variable goes to zero very fast.

REFERENCES

1. Wilson J. Rugh. 1996. *Linear System Theory (2nd Ed.)*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA.
2. Linear System Theory: The State Space Approach (1963) by L.A. Zadeh, C. A Desoer.
3. Kastsuhiko Ogata ,*Modern Control Engineering* , Fourth edition, Pearson Prentice Hall.
4. Leung, F. H F; Tam, P.K.-S.; Li, C.K., "An improved LQR-based controller for switching DC-DC converters," *Industrial Electronics, IEEE Transactions on* , vol.40, no.5, pp.521,528, Oct 1993.
5. Douglas, J.; Athans, M., "Robust LQR Control for the Benchmark Problem," *American Control Conference, 1991* , vol., no., pp.1923,1924, 26-28 June 1991.
6. Datko, Richard, "The LQR Problem for Functional Differential Equations," *American Control Conference, 1993* , vol., no., pp.509,511, 2-4 June 1993.
7. Jones, E., "On the existence of optimal stabilizing controls," *Automatic Control, IEEE Transactions on* , vol.24, no.1, pp.122,124, Feb 1979