PROBLEMS SOLVING USING DIFFERENT REGISTERS OF REPRESENTATION

Elena Fabiola Ruiz Ledesma
National Polytechnic Institute, México
E-mail: efruiz@ipn.mx

Abstract

In this paper are showed on the results of research whose aim was to document and analyze the manner in which students relate different representations when solving problems. A total of 20 students took part in the study, students attending their first year of university studies. In order to design the problem, the underlying information in each representation was deemed to be the starting point of different inferences and of different cognitive processes. The findings obtained make it possible to assert that the underlying information in each representation is not visible to all students and that a problem can foster handling of different representations, the making and verifying of various conjectures and the transfer of knowledge acquired in previous courses.
Key words: *representations, solving problem, verifying conjectures.*

Introduction

A consensus seems to exist among researchers in mathematics education with respect to the importance of relating different representations of a mathematical concept in order to attain the solution to a problem. One example of this is that when students solve a problem that involves the concept of function, they draw a graph or build a table of values to represent it. From the graph or the table it is possible to obtain the information relevant to the function, which can then be employed so as to arrive at the solution of the problem.

Parnafes and Disessa (2004) have expressed themselves along such lines. On the one hand they indicate that student reflections are tied to the representation and context that they are using. They further state that each representation either highlights or hides aspects of a concept, and that when the students use several representations they develop a more flexible understanding of the concept (p. 251). While on the other hand, the same authors state that the relationship among different representations also provides information regarding the cognitive processes of the students in the problem solving process. The underlying information in each representation is the point of departure of different inferences and, consequently, of different cognitive processes (p. 252). The latter idea is the starting point of the research reported on here, which was undertaken with students attending first year of university. The objective of the research is to document and analyze the manner in which the students relate different representations while solving a problem.

Problem of Research

The problem of the research is to review if the relationship among different representations used by the students allows them to identify regularities and relations among the variables in the problem solving process.

Relationship that Exists Among Representations

Since the relationship that exists among representations is the core of the research reported on here, the author of this article believe it is advisable to clarify that in this document the term “to relate representations” is used in the sense of Goldin and Kaput (1996). The foregoing authors point out that “a person relates representations when he is able to integrate his cognitive structures in such a way that given an external representation, the individual is able to predict or identify its counterpart” (p. 416). Several authors (Duval 1998; Parnafes & Disessa, 2004; and Goldin & Kaput 1996) stress the relationship among representations as an important element in stimulating reflection in students. In particular terms, Parnafes and Disessa (2004) refer to use of varying representations in two different ways, to wit: a) use of representations in order to achieve greater understanding of a concept; and, b) use of a representation in order to promote cognitive processes, such as abstraction and generalization.

Use of Representations in Learning a Mathematical Concept

As regards use of representations in comprehension of a concept, Duval (1998) mentions that each representation underlies information concerning the concept. As an example, he alludes to the case of a function represented by a line, in which he indicates that the relevant elements are the direction of the slope of the line, the angles that the line forms with the axis and the position of the line with respect to the origin of the vertical axis. Duval (1998), associates each visual variable with a meaningful unit in algebraic writing. The direction of the line’s slope is related to the sign of coefficient X in the expression $y=ax+b$; the angle formed by the line’s intersection with the axis, to the absolute value of parameter a; and the position of the line with respect to the origin of the vertical axis, to the value of b. According to Duval (1993) adequate knowledge of a concept is considered the invariant of multiple semiotic representations of the concept and it is accomplished when a student skillfully handles changes between representations.

Use of Representations to Make the Cognitive Processes of Students Visible

Parnafes and Disessa (2004) explain that each representation provides specific information and promotes certain cognitive processes. For instance they document the work of students within a computer-aided learning environment in which two representations are used. The first is a representation that simulates the movement of two turtles, while the second is a number representation made up of two lists of values, one of positions and the other of speeds. The two representations characterize the same structure –movement of two objects- yet each one is the starting point of different cognitive processes (p. 252). Based on their analysis of the data, the authors identify two types of reasoning. In one the students arrive at the solution by assigning a value to each variable in the problem, ensuring that all of the restrictions are complied with. In the other, the students create a mental model of the movement of the turtles and analyze those images in order to infer qualitative descriptions from those images.

Digital Technologies and Representations

One can also identify studies that refer to the role of digital technologies and their contribution to having students relate representations. In this respect Goldin and Kaput (1996) state that digital tools, the likes of computers, provide resources for an individual to relate representations. The foregoing is achieved when based on an action in an external representation, the subject is able to predict and identify the results of that action in another representation.

As mentioned by Parnafes and Dissesa (2004), each representation is associated with

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specific information. The problem reported on in the latter document was redesigned so that in addition to using verbal and iconic representations, the students also build a table of values. Usage of an electronic spreadsheet (Excel) can be very useful in this type of activity. Wilson, Ainley and Bills (2004) identify three characteristics of these spreadsheets that may help students to make generalizations and to gain a better understanding of the variables, as follows: a) analysis of the calculations, b) use of notation, and c) the possibility of feedback.

Since the students had used an electronic spreadsheet in three previous activities, there were minimal technical problems related to use of the software and such problems are not cause for thought in this document. Having said this, it would not be good to detract from the importance of the role of digital technologies in student reflections.

Methodology of Research

General Background of Research

For the documentary phase of the research, a review was undertaken of the specialized literature in the fields of semiotic representations (Duval, 1998), the use of representation registers (Hitt, 2005), the importance of technology, the concept of function (Hitt, 2005) and the visualization in the understanding of calculus (Ben-Chaim, 2005; Eisenberg & Dreyfus, 1990, Hitt, 2003 and Zimmermann, 2001).

The methodological orientation stands within a qualitative perspective, which means that the qualitative aspects of the experimental process were fundamentally observed. This was carried out through the following phases:

- Determination of the sample of students to whom the interview would be applied.
- Analysis of or data? of the interviewed
- Discussion
- Determination of results? and conclusions.

Sample of Research

The research was carried out with 10 pairs of first year university students in Mexico City. The work was done with 10 pairs of students to have more control in the activities, besides that the 20 students were part of a group that was taken as a simple. The activity reported on here was undertaken during two sessions of one hour and one half each. The sessions were videotaped and the students submitted their work both in writing and in electronic files.

Instrument and Procedures

Because of the use of a qualitative methodology the procedure consist in an analysis of each referred interviewed in the classroom.

The Problem: The Triangular Stack of Boxes

The problem is proposed by Bassein (1993). It consists of determining how the quantities in a sequence of steps change and of identifying that from the very first step a relationship among the quantities is maintained.

The problem was presented to the students in the following way:

The idea is to build a triangular stack of boxes, as shown in Figure 1.

a) Determine the number of boxes needed to form a stack of a given height.
In this problem the boxes are rectangular and are all the same size.
The height of the stack will be measured in terms of the number of levels in the stack.
For example, a stack made up of three levels will have a height of 3.

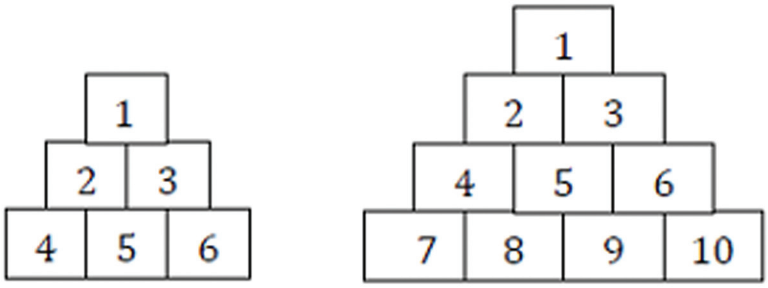

Figure 1: Sorce: Bassein (p.197).

b) You can begin your explorations by drawing stacks made up of different levels.

c) Write down in a table the number of levels and of boxes needed in each drawing.

Two representations are referred to in the problem wording: verbal and iconic. The instruction provided under clause c) suggests to the students that they should make drawings and write down the values represented by different triangular stacks. The clause was included in order to guide the work of the students toward using a numerical representation, and having them prepare a table of values that included the variables of levels and boxes. This was done re-broaching the ideas of Parnafes and Disessa (2004) who indicate that tables of values contribute to facilitating identification by students of patterns and regularities among the series that make up the table, and that they are the basis of particular cognitive processes.

The table of values suggested in clause c) was also included with the intention of orienting the work of the students toward using an algebraic representation. It is important to mention at this point that to obtain a mathematical model that makes it possible to determine the number of boxes needed to build a stack of a given height, it is indeed advisable to work with an algebraic representation.

The students worked on the problem during two sessions. A detailed analysis of their work was subsequently undertaken and this analysis provided information on the manner in which they related the variables present in the problem and the manner in which they developed a mathematical model to represent the relation between them. The actions carried out by the students were followed very closely, paying special attention to their handling of representations.

The questions outlined below arose from the analysis of the data, questions that are the focal point of this document:

- What information do the students obtain from each representation that they use while solving the problem?
- In what manner do the students relate different representations?

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Results of Research

Upon analyzing the data, evidence was identified that enabled us to respond to the questions. The manners of relating representations arose and are exemplified with the work of three students: Emilio, Rodrigo and Abel. The work of the three foregoing students is representative of the work carried out by the remainder of the group. The data used below come from different sources. Brief episodes of the videotaped sessions were taken up again, hence the actions and gestures are explained in parenthesis and pauses are indicated by use of ellipsis.

First Manner. Emilio Related Verbal, Iconic, Numerical and Algebraic Representations.

Emilio read the wording of the problem (verbal representation), drew stacks containing different levels (iconic representation) and carried out a numerical exploration (numerical representation). The instruction to draw stacks with different levels was a contributing factor in his being able to identify a relationship between the height of the stack and the number of boxes in the base level, as is shown in the following episode:

Emilio: …so, analyzing the problem you gave us, the base is equal to the height sought; so we have … if the base is two we add two plus one, then if we have a height of seven, the base is seven.

The drawings done together with the numerical exploration also contributed to Emilio' ability to identify that the total number of boxes needed to create a stack of a given height was equal to the sum of the number of boxes in the base, plus the number of boxes in a stack with one level less, as he explained to the group:

Emilio: … if its height is seven, its base is seven. So finding the total number of boxes is seven, plus six, plus five, plus four, plus three, plus two, plus one *(He places his hand in a horizontal position and raises it to indicate a different level in the stack each time he adds; seven, plus six, plus five, etc.)*

The explanation provided by Emilio attests to his having associated the total number of boxes with an expression that enabled him to calculate the sum of the natural numbers from 1 to n. Emilio was already familiar with this expression.

Emilio: I don't know if you remember, but this formula was given to us by our Physics professor when he asked us what the sum of one though one hundred was. And that was when he gave us the formula, *(writes the formula on the whiteboard)* Figure 2.

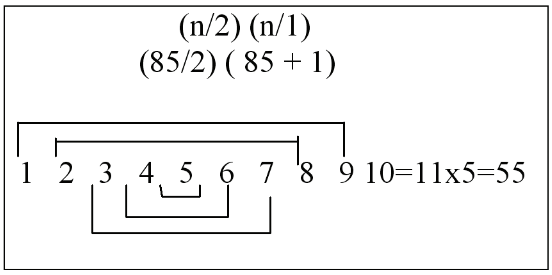

Figure 2: Emilio writes a formula on the whiteboard.

Emilio: So to find the total number of boxes, the base is seven, plus six, plus five, plus four, plus three, plus two, plus one, in other words, all of its previous numbers, and thus we have the formula *(writes the formula on the whiteboard)*, n is the number of levels.

With a few examples, Emilio checked that the formula worked to calculate the number of boxes needed to create a stack of a given height. It is important to point out here that it was not simply a matter of his having memorized a formula; the explanation he gave his classmates and what he wrote down on the whiteboard confirm that he had established a relationship between the numerical representation of the sum of the natural numbers from 1 to n, and the algebraic representation that corresponds to that sum, as can be seen in Figure 2. What is more, Emilio was able to transfer the information he obtained from working with the verbal, iconic and numerical representations to a new context.

Second Manner. Rodrigo Related Verbal, Iconic and Numerical Representations

Rodrigo began his exploration on a sheet of paper, in which he represented particular cases. It is noteworthy to mention that unlike Emilio, Rodrigo counted the number of boxes needed to obtain a stack of a given height. Apparently he did not consider the number of boxes needed to create a stack with one level less. Figure 3 depicts Rodrigo's drawings, submitted in his written report, where he has one stack made up of six levels with twenty-one boxes, one stack made up of seven levels with twenty-eight boxes.

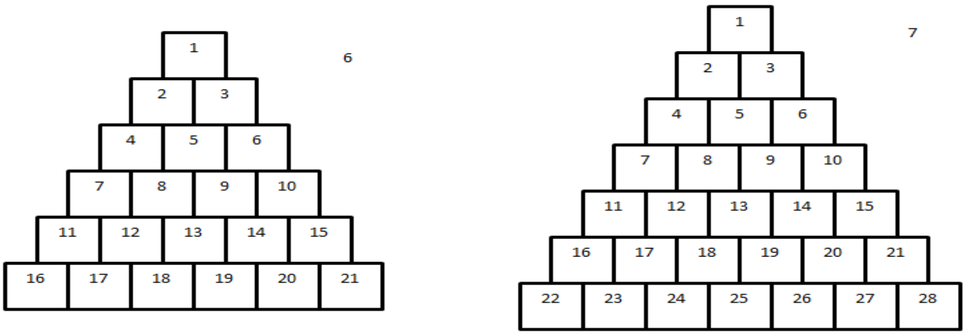

Figure 3: Rodrigo draws particular cases.

In the episode included below, Rodrigo explains to his classmates how he went from the wording of the problem (verbal representation) to the drawings of stacks containing different levels (iconic representation), from which he obtains the information needed to build a table of values (numerical representation).

Rodrigo: *I created a table in Excel (inaudible) and by levels I started with zero …*
Researcher: *You worked directly in Excel?*
Rodrigo: *No (inaudible) first on a sheet of paper (points with his hand to the shape of a triangular stack).*

In figure 4, Rodrigo explains the manner in which he developed a table of values on the electronic spreadsheet. As can be seen during his explanation the student used Excel syntax.

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	A		B
1			
2	0	_______	0
3	1	_______	1
4	2	_______	3
5	3	_______	=B4+A4+1=6
6	4		

Figure 4: Rodrigo explains his work with an electronic spreadsheet.

The episode outlined below includes a transcription of Rodrigo's participation, and represents evidence that the numerical representation contributed to his identification of a relationship among the values in the table. The relation enabled him to determine the number of boxes needed to build a stack of a given height. Nonetheless the operation that he proposes, B4+ A4 + 1, which corresponds to adding the number of boxes needed to create a two-level stack, plus the number of boxes in the base level of that stack, plus one, makes it possible to infer that Rodrigo did not identify the same relation as Emilio did. Emilio pointed out: if its height is seven, its base is seven.

Rodrigo: For one level it's one box: for two levels, 3 boxes... And what I did was: add the two previous ones, plus 1; for example, for 3…, *(He writes the formula =B4+A4+1, and using it he calculates the number of boxes needed to create a three-level stack).*

Rodrigo: So I would be left with three plus two is five, plus one is six. And that's the way I would fill it in up to 50.

In order to encourage Rodrigo to reflect upon the fact the expression he had come up with only worked in particular cases, the researcher intervened as follows:

Researcher: *And what about if you have a stack made up of 75 levels?*
Rodrigo: *I would have to make the list all the way down to 75.*
Rodrigo: *The formula depends on the whole list (Moves his hand upward to indicate the previous numbers in the table).*

Rodrigo found a relation based on his analysis of particular cases, but he was unable to find a general expression to determine the number of boxes needed to build a stack of a given height.

Third Manner. Abel Related Verbal, Iconic, Numerical and Algebraic Representations.

Abel initially worked with the verbal representation, then changed to the iconic representation and drew stacks with different heights. From his analysis of the figures, he obtained information that he used to built a table in Excel. (See table 1). He shows the table he built. Column 1 indicates the number of levels, while column 2 indicates the number of boxes.

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Table 1. Excel table built by Abel.

Number of levels	Number of boxes
1	1
2	3
3	6

The numerical exploration undertaken by Abel in the table facilitated his ability to identify relations between number of levels and number of boxes variables. The extract provided below portrays the analysis he carried out and the explanation he gave to his classmates.

Abel: The way I solved it is you have levels and numbers of boxes: for one level, one box; for two levels, three boxes; then for three levels, six boxes. And here what I found was that this quantity of boxes *(Points to the value three in the number of boxes column)* is equal to the sum of the previous [level] *(Points to the value one in the number of boxes column)*, plus the number of levels that you want to have *(Points to value two in the number of levels column)*.

The work done subsequently by Abel was divided into two sections. In each section one can identify a relation between the number of boxes and number of levels variables.

First Relation Established by Abel

In the extract provided below, Abel proposes an expression to obtain the number of boxes needed to create a stack containing n number of levels.

Abel: I found that the number of boxes is equal to the number of levels that you want to have, plus the number of boxes that you had before, and that gave the result. So I applied the formula to calculate them, equal to … *(*In Figure 5, he has drawn a table on the whiteboard and uses Excel syntax*)* cell B2 + A3 and it gives me a result of 6. So then I just copy and paste *(Uses Excel-related language, copy and paste)*

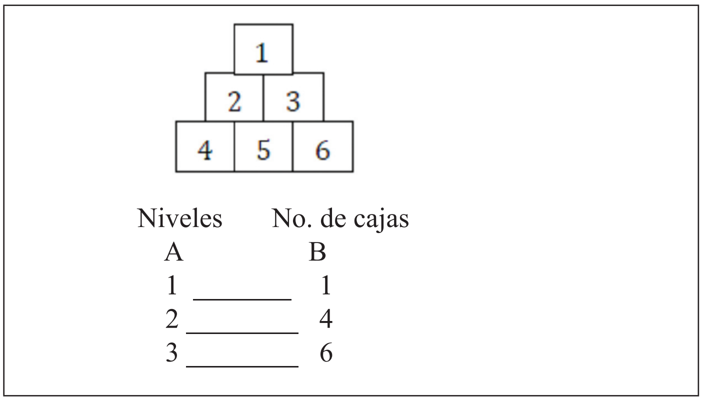

Figure 5: Table built by Abel.

Abel identified that the number of levels in each stack coincided with the number of boxes in the base, he then used that relation to determine the total number of boxes in each stack.

Second Relation Established by Abel

Abel obtained the total number of boxes by adding the number of boxes needed to create a stack with one level less, plus the number of levels from the next stack. In the episode transcribed below Abel explains that in order to obtain the total number of boxes, it is necessary to build a table with a consecutive number of levels.

Abel: The consecutive continued to be maintained *(Points to the column with the number of levels),* but when there was a later one, from 2 to 7 for example, then it didn't give the result, it didn't work. I agreed that this result was correct for consecutive stacks that increased one by one.

Abel: So I began to investigate what relation existed between those two columns *(Point to the number of evels and number of boxes columns),* and thought that this result was a function of this number, with something that could be a sum with something or the product of something, and so I ended up dividing this column *(Points to the number of boxes column)* by this one *(Points to the number of levels column) and* found that this was the result *(Added an additional column, and writes down the result of the divisions).*

Abel: I divided B1 by A1 *(Divides the number of boxes by the number of levels)* which gives me one, then one point five, then two, then two point five, and so on and so forth, point five more each time.

Abel wanted to determine a formula that related the number of levels (column A) with the results of quotients (column C), so as to obtain the number of boxes without having to depend on previous results.

Abel: And then I began to investigate what relationship existed between column A and column C. I knew that the values in column C times the values in column A give the values in column B.

Abel: I cannot depend on these results *(Points to the values in column B)* because that is the result I want to arrive at. So what I did was to see what the relationship between column A and column C was …And I found that it was 3; number of levels divided by 2 plus 0.5; then he does calculations for the remaining values in the table.

Abel: So to be able to arrive at that result, I inputted in the cell the same number as the number of levels, say A1 over 2 plus .5, all of this times A1, and that way it no longer depends on the result of B. So… *(He calculates a result for a 10-level stack).*

$$\left(\frac{10}{2} + .5\right)(10) = 55$$

In the figure 6, one can identify the operations that Abel carried out in Excel.

C8 = =((A8/2)+0.5)*A8

A	B	C	D
NIVELES	No. DE CAJA	((A6/2)+0,5)*A6	(A6/2)*(A6+1)
1	1	1	1
2	3	3	3
3	6	6	6
4	10	10	10
5	15	15	15
6	21	21	21
7	28	28	28
8	36	36	36

Figure 6: Table built by Abel in Excel.

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From the analysis of the data, on was able to find evidence that makes it possible to respond to the questions raised in this document. Each representation provides specific information concerning the problem. The underlying information in each representation is not visible to all of the students, as can be seen in the work of Rodrigo and Abel. The problem put to the students triggered usage of different representations, the making and verification of conjectures, as well as the transfer of knowledge acquired in previous courses, as can be seen in the work done by Emilio.

Use of the numerical representation, which was suggested in the wording of the problem, was a contributing factor in enabling the students to identify regularities and relations among the variables –number of boxes and number of levels- in the manner pointed out by the author of this article. The language used by Emilio and Abel provides proof that they kept the context of the problem in mind at all times. The written reports submitted by Rodrigo, as well as his participation, enable us to infer that he did not relate the numerical representation with the wording of the problem and the figures of the stacks, even when he proposed an expression to calculate the number of boxes needed to build a stack.

Conclusions

The research reported demonstrate the need to coherently utilize different representations that will make it possible to approach the problems more efficiently.

Developing skills linked to mathematics visualization can give students the impetus needed to delve to a deeper level of calculus-related concepts. New materials must absolutely be designed for this integral development, hence refraining from the standard to date of placing much too much emphasis on one single type of representation – the algebraic representation. We must break away from that idea and provide students with a richer notion that will enable them to carry out deeper tasks while they are learning concepts of calculus.

The epistemological approach (disciplinary knowledge) of the course should make it possible to integrate prior structured learning from academic periods at the high school and higher education levels, so that they can be recovered at this level. The objective of the foregoing would be to significantly deal with the concept of variation, including its different techniques, procedures and applications at a level of conceptual depth that makes it possible to raise and solve problems in context, problems that involve important use of algebraic functions as well as the derivatives of those functions.

Engineers take on the tasks of designing and building. And this is why, at the beginning, drawings, graphs and diagrams were a resource inherent in their work. In the training of engineers, use of geometry must be recovered so that the level of visualization they achieve will enable them to develop projects expeditiously.

Endnotes

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Elena Fabiola Ruiz Ledesma	Researcher, The Superior School of Computer Sciences, National Polytechnic Institute, Mexico. Phone: (+5255)57296000 ext. 52041. E-mail: efruiz@ipn.mx