PAMUKKALE ÜNİVERSİTESİ MÜHENDİSLİK FAKÜLTESİ
PAMUKKALE UNIVERSITY ENGINEERING COLLEGE
MÜHENDİSLİK BİLİMLERİ DERGİSİ
JOURNAL OF ENGINEERING SCIENCES
YIL : 1999
CİLT : 5
SAYI : 2-3
SAYFA : 1047-1056

PLASTIC ZONES IN AN INFINITELY LONG TRANSVERSELY ISOTROPIC SOLID CYLINDER CONTAINING A RING-SHAPED CRACK

Mesut UYANER*, Ahmet AKDEMİR*, Seçil ERİM, Ahmet AVCI***
*Selçuk Üniversity, Faculty of Engineering and Architecture, Department of Mechanical Engineering, Konya
**Dokuz Eylül Üniversity, Faculty of Engineering, Department of Mechanical Engineering, Bornova-İzmir

Geliş Tarihi : 06.03.1999

ABSTRACT

In this study, the problem of a ring shaped-crack contained in an infinitely long solid cylinder of elastic perfectly-plastic material is considered. The problem is formulated for a transversely isotropic material by using integral transform technique under uniform load. Due to the geometry of the configuration, Hankel and Fourier integral transform techniques are chosen and the problem is reduced to a singular integral equation. This integral equation is solved numerically by using Gaussian Quadrature Formulae and the values are evaluated for discrete points. The plastic zone lengths are obtained by using the plastic strip model.

Key Words : Transversely isotropic material, Ring-shaped crack, Singular integral equations, Plastic zone

YÜZÜK ŞEKLİNDE ÇATLAKLI SONSUZ UZUNLUKLU ENİNE İZOTROP SİLİNDİRDE PLASTİK BÖLGENİN İNCELENMESİ

ÖZET

Bu çalışmada yüzük şeklinde çatlak bulunduran elastik-ideal plastik sonsuz uzunluklu bir silindir ele alınmıştır. Problem düzgün yayılı yük etkisi altında enine izotrop bir malzeme için integral dönüşüm tekniği kullanılarak formüle edilmiştir. Problemin geometrisi gereğince Hankel ve Fourier integral dönüşüm teknikleri seçilmiş ve problem bir tekil integral denklemine indirgenmiştir. Bu integral denkleminin belli noktalardaki değerleri Gauss Quadrature formülü kullanılarak sayısal olarak elde edilmiştir. Plastik bölge uzunlukları, plastik bant modeli kullanılarak elde edilmiştir.

Anahtar Kelimeler : Enine izotrop malzeme, Yüzük şeklinde çatlak, Tekil integral denklemi, Plastik bölge

1. INTRODUCTION

In the process of designing structural or machine components, one of the important steps is the determination of the final geometry, dimensions of the part and selection of the material in such a way that under given loading and the environmental conditions the part will perform its function properly.

The structural strength of materials generally depends on the material properties, the shape and the size of defects as well as together with the orientation of flaws in the medium. Thus in a design dealing with materials, it is necessary to have a good estimate of the disturbed stress state caused by these flaws.

Fiber reinforced composite materials have been characterized as a transversely isotropic medium having five elastic constants (Jones, 1975; Christiensen, 1979). Hexagonal materials such as magnesium, cadmium and zinc are also transversely isotropic.

There is an increasing interest in anisotropic materials due to high strength over density ratio.

Axially symmetric deformation of a transversely isotropic body of revolution has been studied (Lekhnitskii, 1981).

The distribution of stress in a transversely isotropic cylinder containing penny-shaped crack has been investigated (Parhi and Atsumi, 1975). Dahan (1980; 1981) has searched stress intensity factor and stress distribution in a transversely isotropic solid containing a penny shaped crack. Singular stresses in a transversely isotropic circular cylinder with circumferential edge crack have been examined (Atsumi and Shindo, 1979). Konishi (Konishi, 1972; Konishi and Atsumi, 1973) has studied crack problems in transversely isotropic strip and medium. Fildiş has studied stress intensity factors for an infinitely long transversely isotropic solid cylinder containing a ring shaped cavity (Fildiş, 1991).

In addition to the numerous studies stated above, plastic deformations have also been considered by some researchers. Notably among them Olesiak and Shadley (Olesiak and Shadley, 1969) determined the plastic zone in a thick layer with a disk shaped crack. Crack opening displacements in an orthotropic strip have been found by using the plastic strip model (Kaya and Erdoğan). Plastic deformations in a transversely isotropic layer and cylinder have been studied by Danyluk et all. (Danyluk and Singh, 1985; Danyluk et all., 1991). All the work mentioned above related to plastic studies are based on the Dugdale's hypothesis (Dugdale, 1960). The Dugdale model of a crack in a ductile material was introduced to investigate the inelastic zone at the ends of a stationary slit in steel sheets under static tension. The predictions of Dugdale model agree closely with the experimental results.

In this study, the governing elasticity equation for the transversely isotropic axisymmetric problem in cylindrical coordinates is obtained in terms of a Love type stress function. Hankel and Fourier sine transforms are applied to the stress function because of the geometry of the configuration and boundary conditions. The stress function is expressed in terms of summation of two solutions of the governing equation. Using the boundary conditions, the problem is reduced to a singular integral equation. This singular integral equation is solved by using the Gaussian Quadrature. Then the stress intensity factors at the crack tips are determined. Kaya and Erdogan's method was modified to obtain the plastic zone lengths at the crack tips.

The numerical results have been obtained for various ring-shaped crack sizes. The plastic zone lengths are obtained for axial loading. The results are illustrated by graphs.

2. BASIC FORMULATION

Consider the axisymmetric elasticity problem for a transversely isotropic cylinder shown in Figure 1. The equilibrium and the compatibility equations are expressed by (Lekhnitskii, 1981)

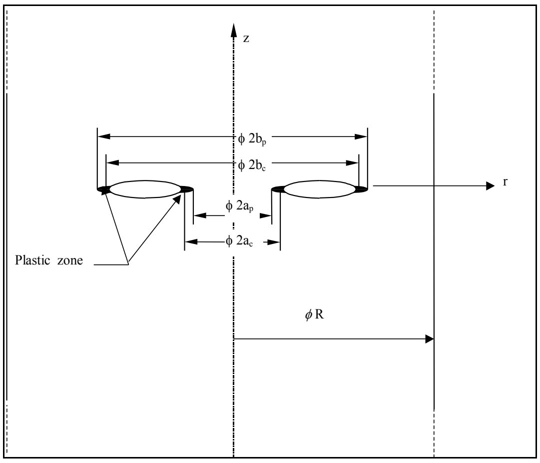

Figure 1. Geometry of the problem

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (2.1.a)$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \qquad (2.1.b)$$

$$\varepsilon_r - \varepsilon_\theta - r\frac{\partial \varepsilon_\theta}{\partial r} = 0 \qquad (2.2.a)$$

$$\frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial z \partial r} = 0 \qquad (2.2.b)$$

For transversely isotropic bodies, axisymmetric deformations can be written as

$$\varepsilon_r = a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z$$

$$\varepsilon_\theta = a_{12}\sigma_r + a_{11}\sigma_\theta + a_{13}\sigma_z \qquad (2.3)$$

$$\varepsilon_z = a_{13}\sigma_r + a_{13}\sigma_\theta + a_{33}\sigma_z$$

$$\gamma_{rz} = a_{44}\sigma_{rz}$$

where

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \quad \varepsilon_\theta = \frac{u_r}{r} \quad \varepsilon_z = \frac{\partial \omega}{\partial z} \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial \omega}{\partial r} \tag{2.4}$$

and a_{11}, a_{12}, a_{13}, a_{33}, and a_{44} are the material constants for the transversely isotropic material.

In terms of stress function $\phi(r, z)$, the stresses may be expressed as

$$\sigma_r = -\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{b}{r}\frac{\partial\phi}{\partial r} + a\frac{\partial^2\phi}{\partial z^2}\right)$$

$$\sigma_\theta = -\frac{\partial}{\partial z}\left(b\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + a\frac{\partial^2\phi}{\partial z^2}\right)$$

$$\sigma_z = \frac{\partial}{\partial z}\left(c\frac{\partial^2\phi}{\partial r^2} + \frac{c}{r}\frac{\partial\phi}{\partial r} + d\frac{\partial^2\phi}{\partial z^2}\right) \tag{2.5}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + a\frac{\partial^2\phi}{\partial z^2}\right)$$

$$u_r = -(1-b)(a_{11}-a_{12})\left(\frac{\partial^2\phi}{\partial r\partial z}\right)$$

$$u_\theta = 0$$

$$w = a_{44}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) + (a_{33}d - 2a_{13}a)\frac{\partial^2\phi}{\partial z^2}$$

where the constants a, b, c, and d are

$$a = \frac{a_{13}(a_{11}-a_{12})}{a_{11}a_{33}-a_{13}^{\;2}}$$

$$b = \frac{a_{13}(a_{13}+a_{44}) - a_{12}a_{33}}{a_{11}a_{33}-a_{13}^{\;2}}$$

$$c = \frac{a_{13}(a_{11}-a_{12}) + a_{44}a_{11}}{a_{11}a_{33}-a_{13}^{\;2}} \tag{2.6}$$

$$d = \frac{a_{11}^{\;2} - a_{12}^{\;2}}{a_{11}a_{33}-a_{13}^{\;2}}$$

For a crack in an infinite cylinder, the problem is an axisymmetric one and the stress function has to be an even function of r. The problem is also symmetric about z = 0 plane and the stress function must be an odd function of z. It is necessary to select a Love type potential function $\phi(r, z)$ of the form the Hankel transform of order zero and Fourier sine transform, that is

$$\phi(r,z) = \int_0^\infty \lambda\left(B_1 e^{m_1\lambda z} + B_2 e^{m_2\lambda z}\right)J_0(\lambda r)d\lambda + \int_0^\infty \frac{2}{\pi}\left(\frac{A.I_0(c_1\alpha r)}{(c_1^2-c_2^2)\alpha^2} + MI_0(c_2\alpha r)\right)\sin(\alpha z)d\alpha \tag{2.7}$$

with

$$m_{1,2} = -\sqrt{\frac{(a+c) \pm \sqrt{(a+c)^2 - 4d}}{2d}}$$

$$c_1 = \sqrt{\frac{2d}{(a+c)+\sqrt{(a+c)^2-4d}}} \tag{2.8}$$

$$c_2 = \sqrt{\frac{(a+c)+\sqrt{(a+c)^2-4d}}{2}}$$

where $J_0(\lambda r)$ and $I_0(c_i\alpha r)$ are the Bessel function of first kind and modified Bessel function of first kind respectively. We find with the help of Eq. 2.5 that

$$\begin{aligned}\sigma_r(r,z) = {} & \int_0^\infty \lambda^4\left[(1-am_1^2)B_1m_1e^{m_1\lambda z} + (1-am_2^2)B_2m_2e^{m_2\lambda z}\right]J_0(\lambda r)d\lambda \\ & - \frac{1-b}{r}\int_0^\infty \lambda^3\left(B_1m_1e^{m_1\lambda z} + B_2m_2e^{m_2\lambda z}\right)J_1(\lambda r)d\lambda \\ & - \frac{2}{\pi}\int_0^\infty \left(\begin{array}{l}\frac{A'}{\alpha^2}\left((c_1^{\;2}-a)\alpha^2 I_0(c_1\alpha r) + \frac{c_1}{r}(b-1)\alpha I_1(c_1\alpha r)\right) \\ + M\left((c_2^{\;2}-a)\alpha^2 I_0(c_1\alpha r) + \frac{c_2}{r}(b-1)\alpha I_1(c_2\alpha r)\right)\end{array}\right)\alpha\cos(\alpha z)d\alpha\end{aligned} \tag{2.9}$$

where

$$A' = \frac{A}{c_1^{\;2} - c_2^{\;2}}$$

$$\begin{aligned}\sigma_{rz}(r,z) = {} & \int_0^\infty \lambda^4\left[\left(1-am_1^{\;2}\right)B_1e^{m_1\lambda z} + \left(1-am_2^{\;2}\right)B_2e^{m_2\lambda z}\right]J_1(\lambda r)d\lambda \\ & + \frac{2}{\pi}\int_0^\infty \left[\left(c_1^3 - c_1a\right)A'\alpha I_1(c_1\alpha r) + \left(c_2^{\;3} - c_2a\right)M\alpha^3 I_1(c_2\alpha r)\right]\sin(\alpha z)d\alpha\end{aligned} \tag{2.10}$$

$$\sigma_z(r,z) = \int_0^\infty \lambda^4 \left[\left(m_1{}^3 d - m_1 c\right) B_1 e^{m_1 \lambda z} + \left(m_2{}^3 d - m_2 c\right) B_2 e^{m_2 \lambda z}\right] J_0(\lambda r) d\lambda$$
$$+ \frac{2}{\pi} \int_0^\infty \left[\left(c_1{}^2 c - d\right) A' \alpha I_0(c_1 \alpha r) + \left(c_2{}^2 c - d\right) M \alpha^3 I_0(c_2 \alpha r)\right] \cos(\alpha z) d\alpha \qquad (2.11)$$

$$\frac{\partial \omega(r,z)}{\partial r} = \int_0^\infty \lambda^4 \begin{bmatrix} (a_{44} - (a_{33} d - 2a_{13} a) m_1^2 B_1 e^{m_1 \lambda z} + \\ (a_{44} - (a_{33} d - 2a_{13} a) m_2^2 B_2 e^{m_2 \lambda z} \end{bmatrix} J_1(\lambda r) d\lambda$$
$$+ \frac{2}{\pi} \int_0^\infty \begin{bmatrix} (a_{44} c_1^2 - a_{33} d + 2a_{13} a) c_1 A' \alpha I_1(c_1 \alpha r) + \\ (a_{44} c_2^2 - a_{33} d + 2a_{13} a) c_2 M \alpha^3 I_1(c_2 \alpha r) \end{bmatrix} \sin(\alpha z) d\alpha \qquad (2.12)$$

3. FORMULATION AND SOLUTION OF THE PROBLEM

Let the ring-shaped crack be embedded in the mid-plane of an infinite cylinder. The material of the cylinder is a transversely isotropic elastic-plastic material. In practice, the curved surface is stress free. Therefore, on the plane z=0, it is required that

$$\sigma_r(R, z) = 0 \qquad 0 < |z| < \infty \qquad (3.1.a)$$

$$\sigma_{rz}(R, z) = 0 \qquad 0 < |z| < \infty \qquad (3.1.b)$$

$$\sigma_{rz}(r, 0) = 0 \qquad 0 \le r \le R \qquad (3.1.c)$$

$$\sigma_z(r, z) = -f(r) \qquad a_p < r < b_p \qquad (3.1.d)$$

$$\omega(r, 0) = 0 \qquad 0 \le r \le a_p \text{ and } b_p \le r \le R \qquad (3.1.e)$$

The cylindrical surface at r = R is free from normal and shear tractions.

We are considering an axially-symmetric deformation of the material and under the Dugdale assumption there are thin annular regions of inelastic deformation surrounding the ring-shaped crack (see Figure 1). The inelastic zones at inner and outer crack tips are described by inner radii a_p and b_c, and outer radii a_c and b_p, respectively. A tensile stress, yield stress of the material, Y is uniformly distributed in the inelastic regions. Therefore, we find that

$$f(r) = \begin{cases} p_0 & a_c < r < b_c \\ -Y & a_p < r < a_c \text{ and } b_c < r < b_p \end{cases} \qquad (3.2)$$

where the pressure p_0 (constant) is prescribed on the crack faces.

Boundary conditions in eq.3.1(a-c) may be used to eliminate three of the four unknowns. The mixed boundary conditions in eq. 3. 1(d-e) may be used to obtain a system of dual integral equations for the fourth unknown function. It is convenient to reduce the mixed boundary condition to an integral equation. The integral equation will be singular. In order to avoid strong singularity in the resulting equation, it is necessary to introduce a new function as the derivative of the displacement ω(r, z), rather than the displacement. The new unknown function will be defined as follows

$$G(r) = \frac{\partial \omega}{\partial r}(r,0) \qquad (3.3)$$

with the help of eq. 2.12, boundary condition 3.1.e and eq. 2.12 are equivalent to

$$G(r) = 0 \qquad 0 \le r \le a_c \qquad b_c \le r \le R \qquad (3.4)$$

$$\int_{a_c}^{b_c} G(r) dr = 0 \qquad (3.5)$$

Substituting Eq. 2.12 into Eq. 3.3 and by using Eq. 3.4, the following equation can be obtained

$$(a_{44} - a_{33} d\, m_1^2 + 2a_{13} a\, m_1^2) B_1 + (a_{44} - a_{33} d\, m_2^2 + 2a_{13} a\, m_2^2) B_2 = H(\lambda) \qquad (3.6)$$

where

$$H(\lambda) = \frac{1}{\lambda^3} \int_{a_c}^{b_c} \rho G(\rho) J_1(\rho \lambda) d\rho \qquad (3.7)$$

by substituting eq.2.10 into eq.3.1.c, we get

$$(1 - a m_1^2) B_1 + (1 - a m_2^2) B_2 = 0 \qquad (3.8)$$

B_1 and B_2 can be easily obtained from eqs. 3.6 and 3.8

$$B_1 = -C_{B_1} H(\lambda) \quad (3.9)$$

$$B_2 = C_{B_2} H(\lambda) \quad (3.10)$$

C_{B1} and C_{B2} are given in appendix.

Now by substituting eq. 2.9 into boundary condition eq. 3.1.a and by taking inverse Fourier-cosine transform,

$$\begin{aligned}&\int_0^\infty \lambda^4 J_0(\lambda R)d\lambda \int_0^\infty [m_1(1-am_1^2)B_1 e^{m_1\lambda z} + m_2(1-am_2^2)B_2 e^{m_2\lambda z}]\cos(\alpha z)dz + \\ &-\frac{1-b}{R}\int_0^\infty \lambda^3 J_1(\lambda R)d\lambda \int_0^\infty [m_1 B_1 e^{m_1\lambda z} + m_2 B_2 e^{m_2\lambda z}]\cos(\alpha z)dz = \frac{A'}{\alpha}C_{11} + M\alpha C_{12}\end{aligned} \quad (3.11)$$

By using the closed form integral (Gradshteyn and Ryzhik, 1980), eq. 3.11 takes the following form

$$F_1 \int_{a_c}^{b_c} \rho G(\rho)d\rho = \frac{A'}{\alpha}C_{11} + \alpha M C_{12} \quad (3.12)$$

Finally by substituting eq. 2.11 into eq. 3.1.b, and by taking inverse Fourier-sine transform,

$$\int_0^\infty \lambda^4 J_1(\lambda R)d\lambda \int_0^\infty [(1-am_1^2)B_1 e^{m_1\lambda z} + (1-am_2^2)B_2 e^{m_2\lambda z}]\sin\alpha z dz = -\left[\frac{A'}{\alpha^2}C_{13} + MC_{14}\right] \quad (3.13)$$

Again by using the closed form integral (Gradshteyn and Ryzhik, 1980), eq. 3.13 takes the following form

$$F_2 \int_{a_c}^{b_c} \rho G(\rho)d\rho = \frac{A'}{\alpha^2}C_{13} + M.C_{14} \quad (3.14)$$

A' and M are obtained from eq.3.12 and eq.3.14

$$A' = \frac{1}{C_2}(C_{14}F_1\alpha - C_{12}F_2\alpha^3)\int_{a_c}^{b_c} \rho G(\rho)d\rho \quad (3.15)$$

$$M = \frac{1}{C_2}(C_{11}F_2\alpha - \frac{1}{\alpha}C_{13}F_1)\int_{a_c}^{b_c} \rho G(\rho)d\rho \quad (3.16)$$

where C_{11}, C_{12} , C_{13}, C_{14} F_1 , F_1 and C_2 are given in Appendix.

Let us substitute A', M, B_1, and B_2 into eq. 2.8.

$$\sigma_z = \left\{\begin{aligned}&\int_0^\infty C_3\rho\lambda J_0(\lambda r)J_1(\lambda\rho)d\lambda \\ &+\frac{2}{\pi}\rho\int_0^\infty \frac{1}{C_2}\left[\begin{aligned}&(c_1^2c-d)\alpha I_0(c_1\alpha r)(C_{14}F_1\alpha - C_{12}F_2\alpha^3) \\ &-(c_2^2c-d)\alpha^2 I_0(c_2\alpha r)(C_{13}F_1\alpha - C_{11}F_2\alpha^3)\end{aligned}\right]\cos\alpha z d\alpha\end{aligned}\right\}\int_{a_c}^{b_c} G(\rho)d\rho \quad (3.17)$$

where

$$C_3 = C_{B_2}(m_2^3 d - m_2 c) - C_{B_1}(m_1^3 d - m_1 c)$$

By substituting eq.3.17 into boundary condition eq.3.1.d, the unknown function $G(\rho)$ can be found as follows

$$\frac{1}{\pi}\int_{a_c}^{b_c}\left[\frac{C_3}{\rho - r} + k(r,\rho)\right]G(\rho)d\rho = -f(r) \tag{3.18}$$

where

$$k(r,\rho) = C_3 k_1(r,\rho) + 2\rho k_2(r,\rho) \tag{3.19}$$

$$k_1(r,\rho) = \frac{m(r,\rho) - 1}{\rho - r} + \frac{m(r,\rho)}{\rho + r} \tag{3.20}$$

$$k_2(r,\rho) = \int_0^\infty \frac{1}{C_2}\begin{bmatrix}(c_1^2 c - d)\alpha I_0(c_1\alpha r)(C_{14}F_1\alpha - C_{12}F_2\alpha^3) \\ -(c_2^2 c - d)\alpha^2 I_0(c_2\alpha r)(C_{13}F_1\alpha - C_{11}F_2\alpha^3)\end{bmatrix}d\alpha \tag{3.21}$$

$$m(r,\rho) = \begin{cases} E\left(\dfrac{r}{\rho}\right) & r < \rho \\ \dfrac{r}{\rho}E\left(\dfrac{\rho}{r}\right) + \dfrac{\rho^2 - r^2}{\rho r}K\left(\dfrac{\rho}{r}\right) & r > \rho \end{cases} \tag{3.22.a}$$

$$m(r,r) = 1 \tag{3.22.b}$$

Here K and E are the first and second kind elliptic integrals respectively.

From the boundary condition 3.1.e and eq. 3.3, it is clear that the integral equation must be solved under the following single valuedness condition.

$$\int_{a_c}^{b_c} G(r)dr = 0 \tag{3.23}$$

For the materials considered in this paper, the numerical values of the modulii c_{ij} (i,j=1..4) are taken from Huntington (Huntington, 1958) and they are tabulated below.

Table 1. Values of Elastic Constants for the Transversely Isotropic Materials (in GPa)

Material	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Magnesium	59.7	26.2	21.7	61.7	16.4
Barium-titanate	168.0	78.0	71.0	189.0	5.46

The elastic constants a_{ij} (i, j = 1..4) can be easily determined by using inverse relation between ε_{ij} and σ_{ij} .

4. NUMERICAL SOLUTION

Examining the kernel in eq. 3.18, when $r = \rho$ it is obvious that the first part of the kernel, $k_1(r, \rho)$ has a simple logarithmic singularity in the form of log│ρ-r│. The second part of the kernel, $k_2(r, \rho)$ is bounded in the closed interval $a \le (r, \rho) \le b$. The unknown function $G(\rho)$ is infinite but integrable at $\rho = \pm 1$, therefore the solution is of the form (Mushkelishvili, 1953).

$$G(\rho) = \Phi(\rho)[(\rho - a)(b - \rho)]^{-1/2} \tag{4.1}$$

A standard numerical technique can be used to find out the unknown function $G(\rho)$ (Erdogan and Gupta, 1972). To be able to apply the numerical solution technique to the singular integral equation, it should be normalized. Normalization is carried out by the following quantities:

$$r = \frac{b_c - a_c}{2}\eta + \frac{b_c + a_c}{2} \tag{4.2.a}$$

$$\rho = \frac{b_c - a_c}{2}\tau + \frac{b_c + a_c}{2} \tag{4.2.b}$$

Eqs. 3.18 and 3.23 become,

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{C_3}{\tau - \eta} + K(\tau,\eta)\right]G(\tau)d\tau = -f(\eta) \tag{4.3}$$

$$\int_{-1}^{1} G(\eta)d\eta = 0 \tag{4.4}$$

where

$$K(\tau,\eta) = \frac{b_c - a_c}{2}k(\tau,\eta) \tag{4.5}$$

Since $G(\tau)$ has an integrable singularity,

$$G(\tau) = (1 - \tau^2)^{-1/2}F(\tau) \tag{4.6}$$

may be written.

The solution of eq.4.3 is determined by using single-valuedness condition in eq. 4.4 (Erdogan and Gupta, 1972).

Substituting eq. 4.6 into eq.4.3 we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{C_3}{\tau - \eta} + K(\tau,\eta)\right]\frac{F(\tau)}{(1 - \tau^2)^{1/2}}d\tau = -f(\eta) \tag{4.7}$$

$F(\tau)$ has to be obtained from eq. 4.7 subjected to the single-valuedness condition,

$$\int_{-1}^{1} \frac{F(\eta)}{(1-\eta^2)^{1/2}} d\eta = 0 \qquad (4.8)$$

Eqs. 4.7 and 4.8 can be evaluated by using the Gauss-Chebyshev integration formula (Scheid, 1968). Thus from Eqs. 4.7 and 4.8 we obtain

$$\sum_{k=1}^{n} \frac{1}{n} F(\tau_k) \left[\frac{C_3}{\tau_k - \eta_r} + K(\tau_k, \eta_r) \right] = -f(\eta_r) \quad (r = 1,..,n-1) \quad (4.9)$$

$$\sum_{r=1}^{n} \frac{\pi}{n} F(\eta_r) = 0 \qquad (4.10)$$

$$\tau_k = \cos\left(\frac{\pi}{2n}(2k-1) \right) \qquad (k = 1,..,n) \qquad (4.11)$$

$$\eta_r = \cos\left(\frac{\pi r}{n} \right) \qquad (r = 1,..,n-1) \qquad (4.12)$$

The set of n simultaneous algebraic equations of 4.9 and 4.10 is solved and one can find n values for $F(\tau_i)$ (i=1..n). In order to determine the stress intensity factors at the inner and outer crack tips, the values of F(+1) and F(-1) must be evaluated from the set of $F(\tau_i)$. Evaluation is performed by means of the interpolation technique (Krenk, 1975).

The Mode I stress intensity factors at the crack tips are defined as

$$k(a_c) = \lim_{r \to a_c} \sqrt{2(a_c - r)}.\sigma_z(r,0) \qquad (4.13)$$

$$k(b_c) = \lim_{r \to b_c} \sqrt{2(r - b_c)}.\sigma_z(r,0) \qquad (4.14)$$

$k(a_c)$ and $k(b_c)$ can also be expressed in terms of unknown function G(r)

$$k(a_c) = \lim_{r \to a_c} C_3 \sqrt{2(a_c - r)}.G(r) = C_3 \sqrt{(b_c - a_c)/2}.F(-1) \qquad (4.15)$$

$$k(b_c) = \lim_{r \to b_c} C_3 \sqrt{2(r - b_c)}.G(r) = -C_3 \sqrt{(b_c - a_c)/2}.F(+1) \qquad (4.16)$$

4. 1. Plastic Zone Sizes

It is considered that, an axially-symmetric deformation of the material and that under the Dugdale assumption, there are thin annular regions of inelastic deformation surrounding the ring-shaped crack tips. A tensile stress **Y** is uniformly distributed in the inelastic regions.

Under the given external loading (represented by $\boldsymbol{\sigma}$) let the plastic zones spread to $r = a_p < a_c$ and $r = b_p > b_c$, a_c and b_c being the initial crack lengths. Solving now the integral equation 3.18 with b_p replacing b_c, for the given external loads, one may obtain a stress intensity factor at b_p. Also a stress intensity factor at a_p may be obtained by solving singular integral equation with a_p replacing a_c. These stress intensity factors would be linearly dependent on the magnitude of external load $\boldsymbol{\sigma}$ and would be functions of b_p/R and a_p/R respectively. Repeating the solutions with only external load $\sigma(0,r) = f(r) = -\mathbf{Y}$ $b_c < r < b_p$ and $a_p < r < a_c$, may again obtain stress intensity factors which would be linearly dependent on **Y** and would be functions of b_c/R and b_p/R, a_c/R and a_p/R, respectively. Here Y is the flow stress and represents the yield behavior of the material. Since the stress state at the fictitious crack tips $r = b_p$ and $r = a_p$ must be bounded, the sum of these two stress intensity factors must be zero satisfying the following conditions (Kaya and Erdogan, 1980).

$$\sigma.k_{1b}(b_p) + Y.k_{2b}(b_p, b_c) = 0 \qquad (4.17.a)$$

$$\sigma.k_{1a}(a_p) + Y.k_{2a}(a_p, a_c) = 0 \qquad (4.17.b)$$

Noting that k_{1a}, k_{1b}, k_{2a}, and k_{2b} correspond to the stress intensity factors calculated from the respective "unit loads". The term $\sigma.k_1$ gives the stress intensity factor under the external load $\boldsymbol{\sigma}$ and $Y.k_2$ gives the stress intensity factor under the flow stress **Y**. Eqs.4.17.a-b provide a simple meaning for calculating the plastic zone sizes b_p-b_c and a_c-a_p for a given "load ratio" $\boldsymbol{\sigma/Y}$ in an inverse manner. It must be emphasized here that at first a load ratio is found by a given plastic zone length b_p-b_c. Then, a plastic zone length a_c-a_p which correspond to the same load ratio must be found with the help of iteration.

5. NUMERICAL RESULTS

The term $(c_1^2 - c_2^2)$ in the denominator of the eq. 2.9 becomes equal to zero for perfectly isotropic materials. Hence the perfectly isotropic materials can not be analyzed by making use of the formulation given in this problem.

Determination of stress intensity factor in magnesium cylinder subjected to unit load, when the crack length is greatest is the one $a_c/R = 0.1$ and $b_c/R = 0.2$ at the crack inner tip, where the first investigation is carried out, the value of $k(a_c)$, as b_c/R reaches to 0.9, increases approximately 6.52 times. This increase at the outer tip of the crack is

3.72 times. Increases in barium-titanate material at the inner and outer tips of the crack, 5.53 and 3.45, respectively. The difference in increase for these two material is 15 %.

The work for determining the length of a developed plastic zone in a various cylinders necessitated solving the singular integral equation 3.18 using a appropriate values of the elastic constants. When the values obtained for the length of the developed plastic zones are examined, it is seen that P_0 / Y ratio in which a plastic zone is formed, is smaller in magnesium while it is bigger in barium-titanate. The difference between the values obtained for each material is maximum 1.5 % and minimum 0.002 %. Therefore, it has been considered adequate only to include the plastic zone variations for magnesium and these variations have been given in the form of graphics in Figs. 2 and 3. Variations of the plastic zone in the cylinder of barium-titanate for $a_c/R = 0.2$, $b_c/R = 0.3$ and $a_c/R = 0.4$, $b_c/R = 0.5$ are given in Figs. 4 and 5 as an example. In plastic zone examinations, the lengths of the plastic zone have been determined by equation 4.17 without calculating the yield stress of the materials. Since the load ratio P_0/Y has been considered instead of yield stress Y, the difference among the plastic zone lengths obtained for each material is very little.

In magnesium cylinder, the variation of the plastic zone length at the outer tip of the crack with the load ratio P_0 / Y for $a_c/R = 0.2$ is given in Figure 2. In this graphic, plastic zone lengths for seven different crack positions have been shown. A constant plastic zone is formed at lower load ratios as the crack length increases. Since the stress intensity factor will increase with the increase of the crack length, P_0/Y ratio will decrease. This result is as expected in connection with the variation of the stress intensity factor along the crack length.

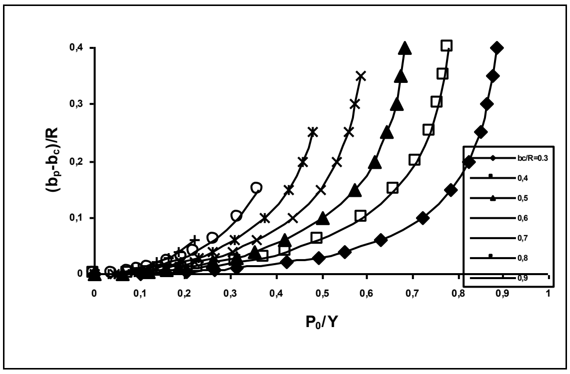

Figure 2.Variation of the plastic zone at outer crack tip in a cylinder of magnesium for $a_c/R = 0.2$

For all series in Figure 2 prior to plastic zone length reaching the cylinder radius R, relatively small increase in the plastic zone length is observed against comparatively large increase in P_0 / Y ratio. As it reaches the cylinder radius R, although the P_0 / Y increases slightly the plastic zone increases rapidly. This cases arises from the fact that, the stress intensity factor increases rapidly as the crack approaches the cylinder outer surface. The length of the final plastic zone length for the series given in graphics, is the length $(R-b_c)$ between the crack outer radius and the cylinder outer radius in which the plastic zone will be greatest for the crack outer tip. Therefore, all of the series has ended at a point approaching this value.

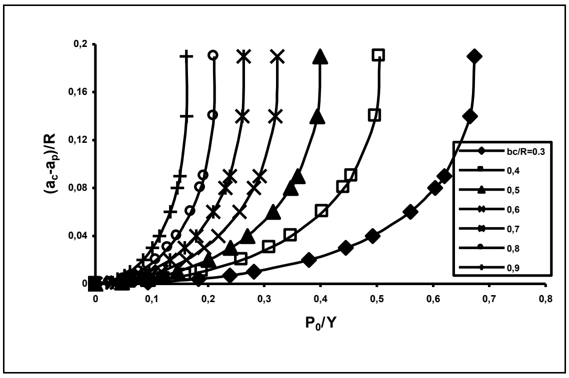

Figure 3. Variation of the plastic zone at inner crack tip in a cylinder of magnesium for $a_c / R = 0.2$

In Figure 3, the variation of the plastic zone length at inner tip of the crack in magnesium cylinder with the load ratio P_0 / Y for $a_c / R = 0.2$ is given. Plastic zone lengths for seven different crack positions are given in this graphic. As observed at the crack outer tip in Figure 2, a constant plastic zone is formed at lower load ratios as the crack length increases. In all series, the plastic zone expands gradually with the increase in P_0/Y ratio and its increase accelerates as the plastic zone length approaches $r = 0$. In the numerical solution of stress intensity factors, the solution goes to infinity for $a_c = 0$. Therefore, as the plastic zone approaches $r = 0$, the stress intensity factor calculated at the end of the plastic zone increases rapidly. This effect can be observed in plastic zone lengths. In the graphics, each of the series has been terminated as it approaches a, where the plastic zone will be greatest.

Another solution observed in the examination of the plastic zone carried out for $a_c / R = 0.2$ is the following: the load ratio P_0 / Y corresponding to any plastic zone at the inner tip of the crack for each b_c / R series, is lower than the load ratio P_0 / Y corresponding to the same plastic zone at outer tip of the crack. For any crack position, the stress intensity factor at the inner tip of the crack is always greater than that at the outer tip of the crack (Fildiş, 1991).

Therefore, since the stress intensity factor at the inner tip of the crack is large, the P_0 / Y ratio is also lower compared to the outer tip of the crack. Greater stress intensity factor causes wider plastic zone. On the contrary, the plastic zone occurring at the inner tip of the crack for any load ratio is greater than that occurring at the outer tip of the crack.

The variations of the plastic zone length at the crack outer and inner tips with the load ratio P_0 / Y in magnesium cylinder for other a_c / R values are similar to that drawn in Figure 2 and 3.

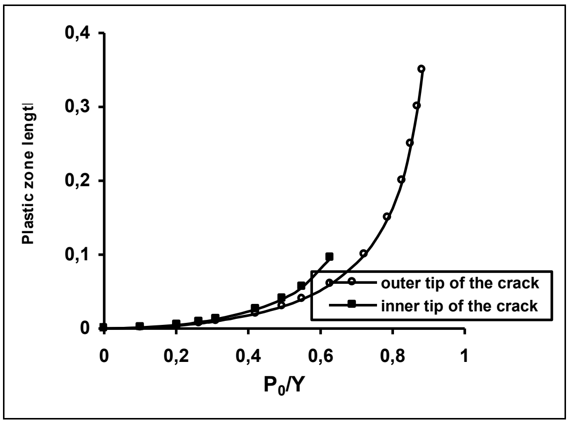

Figure 4. Variation of the plastic zone in a cylinder of barium-titanate for a_c / R = 0.2 and b_c / R = 0.3

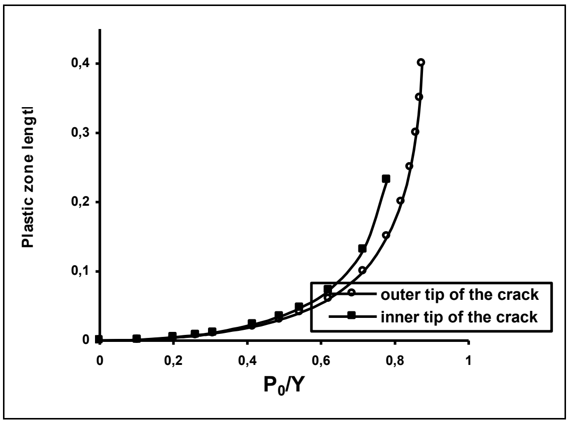

Figure 5. Variation of the plastic zone in a cylinder of barium-titanate for a_c / R = 0.4 and b_c / R = 0.5

The plastic zone investigation for a penny-shaped crack in an infinite transversely isotropic cylinder is given in ref. (Danyluk et all. 1985). In this study graphite-epoxy, E-glass, magnesium, and zinc have been used as materials. The plastic zone length obtained from this study has been made dimensionless by dividing it to crack radius and its variation with the load ratio λ has been presented graphically. Whether the plastic zone expands up to the outer radius of cylinder is not clear from these graphics. In our study, though, this phenomenon can be easily observed.

6. CONCLUSION

In this study, an infinitely long solid cylinder containing a ring shaped crack embedded at its mid plane is investigated under the effect of uniform load. The problem is formulated for a transversely isotropic material by using integral transform technique and then is reduced to a singular integral equation. This integral equation is solved numerically. Plastic zone lengths at the outer and the inner crack tips are determined for various crack configurations.

Depending on the crack geometries, the greatest stress intensity factor was observed in the magnesium cylinder. On the contrary, the greatest plastic zone length develops in barium-titanate cylinder. The largest increase in stress intensity factors is observed in magnesium, the difference between the magnesium and barium-titanate in which smaller increase is observed, being 15 %. This difference between the stress intensity factors is not met however in plastic zone lengths where the largest value in approximately 1.5 %. The small difference between the plastic zones compared to the relatively large difference in stress intensity factors, arises from the fact that plastic zones are determined by the load ratio P_0/Y instead of P_0.

7. APPENDIX

$$C_{B_1} = \frac{1 - am_2^2}{(m_1^2 - m_2^2)(a_{33}d - 2a_{13}a - a_{44}a)}$$

$$C_{B_2} = \frac{1 - am_1^2}{(m_1^2 - m_2^2)(a_{33}d - 2a_{13}a - a_{44}a)}$$

$$C_{11} = (c_1^2 - a)\alpha^2 I_0(c_1\alpha R) + \frac{c_1\alpha}{R}(b-1)I_1(c_1\alpha R)$$

$$C_{12} = (c_2^2 - a)\alpha^2 I_0(c_2\alpha R) + \frac{c_2\alpha}{R}(b-1)I_1(c_2\alpha R)$$

$$C_{13} = (c_1^3 - c_1 a)\alpha^3 I_1(c_1\alpha R)$$

$$C_{14} = (c_2^3 - c_2 a)\alpha^3 I_1(c_2\alpha R)$$

$$C_2 = C_{11}C_{14} - C_{12}C_{13}$$

$$F_1 = C_{B_1}\frac{\alpha}{m_1}(1-am_1^2)I_1\left(\frac{\rho\alpha}{m_1}\right)K_0\left(\frac{R\alpha}{|m_1|}\right)+C_{B_2}\frac{\alpha}{m_2}(1-am_2^2)I_1\left(\frac{\rho\alpha}{m_2}\right)K_0\left(\frac{R\alpha}{|m_2|}\right)$$
$$+\frac{1-b}{R}\left[C_{B_1}I_1\left(\frac{\rho\alpha}{m_1}\right)K_1\left(\frac{R\alpha}{|m_1|}\right)+C_{B_2}I_1\left(\frac{\rho\alpha}{m_2}\right)K_1\left(\frac{R\alpha}{|m_2|}\right)\right]$$

$$F_2 = C_{B_1}\frac{(1-am_1^2)}{m_1^2}I_1\left(\frac{\rho\alpha}{m_1}\right)K_1\left(\frac{R\alpha}{|m_1|}\right)-C_{B_2}\frac{(1-am_2^2)}{m_2^2}I_1\left(\frac{\rho\alpha}{m_2}\right)K_1\left(\frac{R\alpha}{|m_2|}\right)$$

8. REFERENCES

Atsumi, A., Shindo, Y. 1979. Singular Stresses in a Transversely Isotropic Circular Cylinder with Circumferential Edge Crack. Int. J. Engng Sci.. 17, 1229-1236.

Christensen, R. M. 1979. Mechanics of Composite Materials. John Wiley & Sons, New York.

Dahan, M. 1980. Stress Intensity Factor for an Infinite Transversely Isotropic Solid Containing a Penny-Shaped Crack, J. Physique. 41, 213-215.

Dahan, M. 1981. Stress Distribution in a Transversely Isotropic Solid Containing a Penny-shaped Crack. Arch. Mech. 33 (3), 415-428.

Danyluk, H. T. and Singh, B.M. 1985. Plastic Deformation in a Thick Transversely Isotropic Layer Containing a Penny-Shaped Crack. Acta Mechanica. 55, 75-92..

Danyluk, H. T., Singh, B. M., Vrbik, J. 1991. Ductile Penny-shaped Crack in a Transversely Isotropic Cylinder. Int. J. of Fracture. 51, 331-342.

Dugdale, D. S. Yielding of Steel Sheets Containing Slits. 1960. J. Mech. Phys. Solids. 8, 100-104.

Erdogan, F., Gupta, G. D. 1972. On the Numerical Solution of Singular Integral Equations. Quarterly of Applied Math. 525-534, January 1972.

Fildiş, H. 1991. The Stress Intensity Factors for an Infinitely Long Transversely Isotropic Solid Cylinder Which Contains A Ring-Shaped Cavity. MS Thesis in Mechanical Engineering, Middle East Technical University, Ankara.

Gradshyteyn, I. S., Ryzhik, I. M. 1980. Table of Integrals, Series, and Products. Academic Press. Inc., New York.

Huntington, H. B. 1958. The Elastic Constants of Crystals. In: Solid State Physics, Advances in Research and Development, 7, F. Seitz and D. Turnbull (eds.), Academic Press., pp.231-351, New York.

Jones, R. M.. 1975. Mechanics of Composite Materials. McGraw-Hill, Kogakusha Ltd., Tokyo.

Kaya, A. C., Erdogan, F. 1980. Stress Intensity Factors and COD in an Orthotropic Strip. Int. J. of Fracture, 16(2), 171-190.

Konishi, Y., Atsumi, A. 1973. Crack Problem of Transversely Isotropic Strip. Int. J. Engng Sci., 11, 9-20.

Konishi, Y. 1972. On Two Coplanar Cracks in an Infinite Transversely Isotropic Medium, Int. J. Engng Sci., 10, 917-923.

Krenk, S. 1975. A Note on the Use of Interpolation Polynomial Solution of Singular Integral Equations, Quarterly Applied Mathematics, 32, 479-485.

Lekhnitskii, S. G. 1981. Theory of Elasticity of an Anisotropic Body. Mir Publishers, Moscow.

Mushkelishvili, N. I. 1953. Singular Integral Equations. Nordhoof, Groningen, Netherlands.

Olesiak, Z., Shadley, J. R. 1969. Plastic Zone in a Thick Layer with a Disc-Shaped Crack, Int. J. of Fracture Mech. 5(4), 305-313.

Parhi, H., Atsumi, A. 1975. The Distribution of Stress in a Transversely Isotropic Cylinder Containing a Penny-shaped Crack. Int. J. Engng. Sci., 4, 675-685.

Scheid, F. 1968. Theory and Problems of Numerical Analysis. pp. 34, 79. Schaum's Outline Series, McGraw-Hill Book Co., New York.