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Some Recurrences for Generalzed Hypergeometric Functions

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ABSTRACT: In the present paper our result is the q-extension of the known result due to Galu´e and Kalla [4]. Which are define for Generalized hypergeometric function ${}_{s+1}F_s(\cdot)$ in termsof an iterated q-integrals involving Gauss's hypergeometric function ${}_2F_1(\cdot)$. By using the relations between q-contiguous hypergeometric function ${}_2F_1(\cdot)$, some new & known recurrence relations for the generalized hypergeometric functions of one variable are deduced in the line of Purohit [6] as a special case.

I. INTRODUCTION

The generalized basic hypergeometric series Gasper and Rahman [5] is given by

$${}_r\emptyset_s(a_1,\dots,a_r; b_1,\dots,b_s; x) = {}_r\emptyset_s\left[\begin{matrix} a_1, a_2, \dots a_r; \\ b_1, b_2, \dots b_s; \end{matrix}\; q, x\right]$$

$$= {}_r\emptyset_s[(a_r);(b_s); q, x] = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n} x^n \{(-1)^n q^{n(n-1)/2}\}^{(1+s-r)} \qquad \dots(1.1)$$

where for real or complex a

$$(a;q)_n = \begin{cases} 1 & ; \text{ if } n = 0 \\ (1-a)(1-aq)\dots\dots(1-aq^{n-1}) & ; \text{ if } n \in N \end{cases} \qquad \dots(1.2)$$

is the q-shifted factorial, r and s are positive integers, and variable x, the numerator parameters $a_1,\dots,a_r$, and the denominator parameters $b_1,\dots,b_s$ being any complex quantities provided that
$b_j = q^{-m}$, $m = 0,1,\cdots;\ j = 1,2,\cdots,s$.

If $|q| < 1$, the series (1.1) converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|x| < |b_1,\dots\dots,b_s| / |a_1,\dots\dots,a_r|$.

Further, in terms of the q-gamma function, (1.2) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)\,(1-q)^n}{\Gamma_q(a)}, \quad n > 0, \qquad \dots(1.3)$$

where the q-gamma function (Gasper and Rahman [5]) is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{n-1}} = \frac{(q;q)_{n-1}}{(1-q)^{n-1}} \qquad \dots(1.4)$$

where $a \neq 0,-1,-2,\cdots$.

Gasper and Rahman [5], has given the following relations between q-contigous basic hypergeometric functions

$$ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma-1}; \end{matrix} q, x\right] - {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] = q^{\gamma} x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})}\, {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}; \\ q^{\gamma+1}; \end{matrix} q, x\right] \quad \ldots(1.5) $$

$$ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] - {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] = q^{\alpha} x \frac{(1-q^{\beta})}{(1-q^{\gamma})}\, {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}; \\ q^{\gamma+1}; \end{matrix} q, x\right] \quad \ldots(1.6) $$

$$ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta}; \\ q^{\gamma+1}; \end{matrix} q, x\right] - {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] = q^{\alpha} x \frac{(1-q^{\gamma-\alpha})(1-q^{\beta})}{(1-q^{\gamma+1})(1-q^{\gamma})}\, {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1}; \\ q^{\gamma+2}; \end{matrix} q, x\right] \quad \ldots(1.7) $$

and

$$ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta-1}; \\ q^{\gamma}; \end{matrix} q, x\right] - {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] = q^{\alpha} x \frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})}\, {}_2\emptyset_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta}; \\ q^{\gamma+1}; \end{matrix} q, x\right] \quad \ldots(1.8) $$

The following results are due to Galu´e and Kalla [4],

$$ {}_{s+1}\emptyset_s\left[\begin{matrix} \alpha, \beta, a_3, a_4, \ldots, a_s, a_{s+1}; \\ \gamma, b_2, b_3, \ldots, b_s; \end{matrix} x\right] = \sum_{i=0}^{s-2} \Gamma\left[\begin{matrix} b_{s-i} \\ a_{s+1-i},\ b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \int_0^1 \int_0^1 \ldots\ldots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (1-t_{i+1})^{b_{s-i}-a_{s+1-i}-1} $$

$$ \times\ {}_2\emptyset_1\left[\begin{matrix} \alpha, \beta; \\ \gamma; \end{matrix} t_{s-1}\ldots\ldots t_2 t_1 x\right] dt_{s-1} \ldots\ldots dt_2\, dt_1 \quad \ldots(1.9) $$

where Re(b_{s-i}) > 0, for all i = 0, 1, · · · , s-2 , $|x| < 1$ and $|t_{s-1}\ldots\ldots t_2 t_1 x| < 1$.

In view of limit formulae $\lim_{q\to 1^-} \Gamma_q(a) = \Gamma(a)$ and $\lim_{q\to 1^-} \frac{(q^{a};q)_n}{(1-q)^n} = (a)_n$ … (1.10)

where $(a)_n = a\,(a+1)\ \ldots\ldots\ldots\ldots\ (a+n-1)$,

Due to Gasper and Rahman [5], the q-analogue of Euler's integral representation

$$ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha}, q^{\beta}; \\ q^{\gamma}; \end{matrix} q, x\right] = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \times \int_0^1 t^{b-1} (tq;q)_{c-b-1} \times {}_1\emptyset_0\left[\begin{matrix} q^{\alpha}; \\ -; \end{matrix} q, tx\right] d_q t \quad \ldots(1.11) $$

The generalization of q-analogue of Euler's integral representation due to Gasper and Rahman [5] is

$$ {}_r\emptyset_s\left[\begin{matrix} a_1, a_2, a_3, a_4, \ldots, a_s, a_{s+1}; \\ b_1, b_2, b_3, \ldots, b_s; \end{matrix} q, x\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1} \end{matrix}\right] \times \int_0^1 t^{a_r-1} (tq;q)_{b_s-a_r-1} $$

$$ \times\ {}_{r-1}\emptyset_{s-1}\left[\begin{matrix} a_1, a_2, a_3, \ldots a_{r-1}; \\ b_1, b_2, b_3, \ldots b_{s-1}; \end{matrix} q, tx\right] d_q t \quad \ldots(1.12) $$

Purohit [6], has given the iterated q-integral representaion

$$ {}_{s+1}\emptyset_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \ldots, a_s, a_{s+1}; \\ q^{\gamma}, b_2, b_3, \ldots, b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i},\ b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \int_0^1 \int_0^1 \ldots\ldots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1}q;q)_{b_{s-i}-a_{s+1-i}-1} $$

$$\times\ {}_2\emptyset_1\left[\begin{matrix} q^{\alpha},\ q^{\beta}; \\ q^{\gamma}; \end{matrix} q, t_{s-1} \ldots\ldots t_2 t_1 x\right] d_q t_{s-1} \ldots\ldots d_q t_2 d_q t_1. \qquad \ldots (1.13)$$

where $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdot\cdot\cdot, s-2$; $|q| < 1$, $|x| < 1$ and $|t_{s-1} \ldots\ldots t_2 t_1 x| < 1$.

Bailey [1], Exton [3], Galu´e and Kalla [4], Gasper and Rahman [5] & Slater [8], has given a wide range of applications of the theory of generalized hypergeometric functions of one and more variables in various fields of Mathematics, Physics and Engineering Sciences, namely-Number theory, Partition theory, Combinatorial analysis, Lie theory, Fractional calculus, Integral transforms, Quantum theory etc.

In the present article, Generalized hypergeometric function ${}_{s+1}F_S(\cdot)$ is expressed in terms of an iterated q-integrals involving the q-Gauss hypergeometric function ${}_2F_1(\cdot)$. By using q-contiguous relations for ${}_2F_1(\cdot)$, some recurrence relations for the generalized hypergeometric functions of one variable are obtained in the line of Purohit [6] as a special case. Here the mentioned technique is a q-version of the technique used by Galu´e and Kalla [4].

II. MAIN RESULT

If $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdot\cdot\cdot, s-2$ and $|q| < 1$, then the iterated q-integral representaion of ${}_{s+1}F_S(\cdot)$ is given by

$${}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r},\ a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r},\ b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i},\ b_{s-i} - a_{s+1-i} \end{matrix}\right]$$

$$\times \int_0^1 \int_0^1 \ldots\ldots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1}$$

$$\times\ {}_2F_1\left[\begin{matrix} q^{a+r},\ q^{b+r}; \\ q^{c+r}; \end{matrix} q, t_{s-1} \ldots\ldots t_2 t_1 x\right] d_q t_{s-1} \ldots\ldots d_q t_2 d_q t_1. \qquad \ldots(2.0)$$

where $|x| < 1$, $r > 0$ and $|t_{s-1} \ldots\ldots t_2 t_1 x| < 1$.

Proof: Using equation (1.12),the L.H.S. of equation(2.0) can also be written as

$${}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r},\ a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r},\ b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1} \end{matrix}\right] \times \int_0^1 t_1^{a_{s+1}-1} (t_1 q; q)_{b_s - a_{s+1} - 1}$$

$$\times\ {}_sF_{s-1}\left[\begin{matrix} q^{a+r}, q^{b+r},\ a_3, \ldots a_s; \\ q^{c+r},\ b_2, b_3, \ldots b_{s-1}; \end{matrix} q, t_1 x\right] d_q t_1. \qquad \ldots(2.1)$$

repeating the process in the right-hand side of (2.1), we have

$${}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r},\ a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r},\ b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1} \end{matrix}\right] \times \Gamma_q\left[\begin{matrix} b_{s-1} \\ a_s, b_{s-1} - a_s \end{matrix}\right]$$

$$\times \int_0^1 \int_0^1 t_2^{a_s-1} (t_2 q; q)_{b_{s-1} - a_s - 1}\ t_1^{a_{s+1}-1} (t_1 q; q)_{b_s - a_{s+1} - 1}$$

$$\times\ {}_{s-1}F_{s-2}\left[\begin{matrix} q^{a+r}, q^{b+r},\ a_3, \ldots a_{s-1}; \\ q^{c+r},\ b_2, b_3, \ldots b_{s-2}; \end{matrix} q, t_2 t_1 x\right] d_q t_2 d_q t_1. \qquad \ldots(2.2)$$

again repeating the process in the right-hand side of (2.2), we have

$$
{}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \dots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1}\end{matrix}\right] \times \Gamma_q\left[\begin{matrix} b_{s-1} \\ a_s, b_{s-1} - a_s\end{matrix}\right]
$$

$$
\times \Gamma_q\left[\begin{matrix} b_{s-2} \\ a_{s-1}, b_{s-2} - a_{s-1}\end{matrix}\right] \times \int_0^1 \int_0^1 \int_0^1 t_3^{a_{s-1}-1} (t_3 q; q)_{b_{s-2}-a_{s-1}-1}\, t_2^{a_s-1}
$$

$$
\times (t_2 q; q)_{b_{s-1}-a_s-1}\, t_1^{a_{s+1}-1} (t_1 q; q)_{b_s-a_{s+1}-1}
$$

$$
\times {}_{s-2}F_{s-3}\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, \dots a_{s-2}; \\ q^{c+r}, b_2, b_3, \dots b_{s-3}; \end{matrix} q, t_3 t_2 t_1 x\right] d_q t_3\, d_q t_2\, d_q t_1 \qquad \dots (2.3)
$$

On successive operations (s − 4) times in the right-hand side of (2.3), we get

$$
{}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \dots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i}\end{matrix}\right]
$$

$$
\times \int_0^1 \int_0^1 \dots \dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1}
$$

$$
\times {}_2F_1\left[\begin{matrix} q^{a+r}, q^{b+r}; \\ q^{c+r}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x\right] d_q t_{s-1} \dots\dots d_q t_2\, d_q t_1 \qquad \dots (2.4)
$$

which is the required result.

III. RECURRENCE RELATIONS

Using equation (1.5), we get

$$
{}_2F_1\left[\begin{matrix} q^{a+r}, q^{b+r}; \\ q^{c+r}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x\right] = {}_2F_1\left[\begin{matrix} q^{a+r}, q^{b+r}; \\ q^{c+r-1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x\right]
$$

$$
- q^{c+r} t_{s-1} \dots \dots t_2 t_1 x \frac{(1-q^{a+r})(1-q^{b+r})}{(q-q^{c+r})(1-q^{c+r})}\, {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}; \\ q^{c+r+1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x\right] \qquad \dots (3.1)
$$

On substituting value from relation (3.1) in the right-hand side of the equation (2.0), we have

$$
{}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \dots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i}\end{matrix}\right]
$$

$$
\times \int_0^1 \int_0^1 \dots \dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1}
$$

$$
\times {}_2F_1\left[\begin{matrix} q^{a+r}, q^{b+r}; \\ q^{c+r-1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x\right] d_q t_{s-1} \dots\dots d_q t_2\, d_q t_1
$$

$$
- q^{c+r} t_{s-1} \dots \dots t_2 t_1 x \frac{(1-q^{a+r})(1-q^{b+r})}{(q-q^{c+r})(1-q^{c+r})} \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i}\end{matrix}\right]
$$

$$\times \int_0^1 \int_0^1 \dots \dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q ; q)_{b_{s-i}-a_{s+1-i}-1}$$

$$\times {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r+1}; \\ & q^{c+r+1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right] d_q t_{s-1} \dots \dots d_q t_2 d_q t_1 . \quad \dots(3.2)$$

Again, on making use of the result (2.0), the above result (3.2) leads to the following recurrence relation:

$${}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, & a_3, a_4, \dots a_s, a_{s+1} ; \\ & q^{c+r}, \ b_2, b_3, \dots b_s ; \end{matrix} q, x \right] = {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, & a_3, a_4, \dots a_s, a_{s+1} ; \\ & q^{c+r-1}, \ b_2, b_3, \dots b_s ; \end{matrix} q, x \right]$$

$$- q^{c+r} x \frac{(1-q^{a+r})(1-q^{b+r})}{(q-q^{c+r})(1-q^{c+r})} \sum_{i=0}^{s-2} \left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})} \right]$$

$$\times {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, \ q^{b+r+1}, & a_3 q, a_4 q, \dots \dots, a_s q, a_{s+1} q ; \\ & q^{c+r+1}, \ b_2 q, b_3 q, \dots \dots, b_s q ; \end{matrix} q, x \right] \quad \dots(3.3)$$

where $\mathrm{Re}(b_{s-i}) > 0$, for all i = 0, 1, · · · , s− 2 and |x| < 1.

Using equation (1.6), we get

$${}_2F_1\left[\begin{matrix} q^{a+r}, & q^{b+r}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right] = {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right]$$

$$- q^{a+r} t_{s-1} \dots \dots t_2 t_1 \, x \frac{(1-q^{b+r})}{(1-q^{c+r})}$$

$$\times {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r+1}; \\ & q^{c+r+1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right] \quad \dots(3.4)$$

On substituting value from equation (3.4) in the right-hand side of the equation (2.0), we have

$${}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, & a_3, a_4, \dots a_s, a_{s+1} ; \\ & q^{c+r}, \ b_2, b_3, \dots b_s ; \end{matrix} q, x \right] = \sum_{i=0}^{s-2} \Gamma_q \left[\begin{matrix} & b_{s-i} \\ a_{s+1-i}, & b_{s-i} - a_{s+1-i} \end{matrix}\right]$$

$$\times \int_0^1 \int_0^1 \dots \dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q ; q)_{b_{s-i}-a_{s+1-i}-1}$$

$$\times {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right] d_q t_{s-1} \dots \dots d_q t_2 d_q t_1$$

$$- q^{a+r} t_{s-1} \dots \dots t_2 t_1 \, x \frac{(1-q^{b+r})}{(1-q^{c+r})} \sum_{i=0}^{s-2} \Gamma_q \left[\begin{matrix} & b_{s-i} \\ a_{s+1-i}, & b_{s-i} - a_{s+1-i} \end{matrix}\right]$$

$$\times \int_0^1 \int_0^1 \dots \dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q ; q)_{b_{s-i}-a_{s+1-i}-1}$$

$$\times {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r+1}; \\ & q^{c+r+1}; \end{matrix} q, t_{s-1} \dots \dots t_2 t_1 x \right] d_q t_{s-1} \dots \dots d_q t_2 d_q t_1 . \quad \dots(3.5)$$

Again, on making use of the result (2.0), the above result (3.5) leads to the following recurrence relation:

$$ {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, q^{b+r}, a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \ldots b_s; \end{matrix} q, x\right] $$

$$ - q^{a+r} x \frac{(1-q^{b+r})}{(1-q^{c+r})} \sum_{i=0}^{s-2} \left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] $$

$$ \times {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}, a_3 q, a_4 q, \ldots, a_s q, a_{s+1} q; \\ q^{c+r+1}, b_2 q, b_3 q, \ldots, b_s q; \end{matrix} q, x\right] \quad \ldots (3.6) $$

where Re(b_{s-i}) > 0, for all i = 0, 1, · · · , s− 2 and |x| < 1.

Using equation (1.7), we get

$$ {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}; \\ q^{c+r+1}; \end{matrix} q, t_{s-1} \ldots \ldots t_2 t_1 x\right] = {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r}; \\ q^{c+r+1}; \end{matrix} q, t_{s-1} \ldots \ldots t_2 t_1 x\right] $$

$$ - q^{a+r} t_{s-1} \ldots \ldots t_2 t_1 x \frac{(1-q^{c-a})(1-q^{b+r})}{(1-q^{c+r+1})(1-q^{c+r})} $$

$$ \times {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}; \\ q^{c+r+2}; \end{matrix} q, t_{s-1} \ldots \ldots t_2 t_1 x\right] \quad \ldots (3.7) $$

On substituting value from equation (3.7) in the right-hand side of the equation (2.0), we have

$$ {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \int_0^1 \int_0^1 \ldots \ldots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1} $$

$$ \times {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r}; \\ q^{c+r+1}; \end{matrix} q, t_{s-1} \ldots \ldots t_2 t_1 x\right] d_q t_{s-1} \ldots \ldots d_q t_2 d_q t_1 $$

$$ - q^{a+r} t_{s-1} \ldots \ldots t_2 t_1 x \frac{(1-q^{c-a})(1-q^{b+r})}{(1-q^{c+r+1})(1-q^{c+r})} \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times {}_2F_1\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}; \\ q^{c+r+2}; \end{matrix} q, t_{s-1} \ldots \ldots t_2 t_1 x\right] d_q t_{s-1} \ldots \ldots d_q t_2 d_q t_1. \quad \ldots (3.8) $$

Again, on making use of the result (2.0), the above result (3.8) leads to the following recurrence relation:

$$ {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r}, b_2, b_3, \ldots b_s; \end{matrix} q, x\right] = {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, q^{b+r}, a_3, a_4, \ldots a_s, a_{s+1}; \\ q^{c+r+1}, b_2, b_3, \ldots b_s; \end{matrix} q, x\right] $$

$$ - q^{a+r} x \frac{(1-q^{c-a})(1-q^{b+r})}{(1-q^{c+r+1})(1-q^{c+r})} \sum_{i=0}^{s-2} \left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] $$

$$ \times {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, q^{b+r+1}, a_3 q, a_4 q, \ldots, a_s q, a_{s+1} q; \\ q^{c+r+2}, b_2 q, b_3 q, \ldots, b_s q; \end{matrix} q, x\right] \quad \ldots (3.9) $$

Where, Re(b_{s-i}) > 0, for all i = 0, 1, · · · , s− 2 and |x| < 1.

Using equation (1.8), we get

$$ {}_2F_1\left[\begin{matrix} q^{a+r}, & q^{b+r}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots\dots t_2 t_1 x\right] = {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r-1}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots\dots t_2 t_1 x\right] $$

$$ - q^{a+r} t_{s-1} \dots\dots t_2 t_1 x \frac{(1-q^{b-a+1})}{(1-q^{c+r})} $$

$$ \times\ {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r}; \\ & q^{c+r+1}; \end{matrix} q, t_{s-1} \dots\dots t_2 t_1 x\right] \qquad \dots(3.10) $$

On substituting value from equation (3.10) in the right-hand side of the equation (2.0), we have

$$ {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, & a_3, a_4, \dots, a_s, a_{s+1}; \\ & q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] = \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} & b_{s-i} \\ a_{s+1-i}, & b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \int_0^1 \int_0^1 \dots\dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1} $$

$$ \times\ {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r-1}; \\ & q^{c+r}; \end{matrix} q, t_{s-1} \dots\dots t_2 t_1 x\right] d_q t_{s-1} \dots\dots d_q t_2 d_q t_1 $$

$$ - q^{a+r} t_{s-1} \dots\dots t_2 t_1 x \frac{(1-q^{b-a+1})}{(1-q^{c+r})} \quad \sum_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} & b_{s-i} \\ a_{s+1-i}, & b_{s-i} - a_{s+1-i} \end{matrix}\right] $$

$$ \times \int_0^1 \int_0^1 \dots\dots (s-1)\text{times} \sum_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1} q; q)_{b_{s-i}-a_{s+1-i}-1} $$

$$ \times\ {}_2F_1\left[\begin{matrix} q^{a+r+1}, & q^{b+r}; \\ & q^{c+r+1}; \end{matrix} q, t_{s-1} \dots\dots t_2 t_1 x\right] d_q t_{s-1} \dots\dots d_q t_2 d_q t_1. \quad \dots(3.11) $$

Again, on making use of the result (2.0), the above result (3.11) leads to the following recurrence relation:

$$ {}_{s+1}F_s\left[\begin{matrix} q^{a+r}, q^{b+r}, & a_3, a_4, \dots, a_s, a_{s+1}; \\ & q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] = {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, q^{b+r-1}, & a_3, a_4, \dots, a_s, a_{s+1}; \\ & q^{c+r}, b_2, b_3, \dots, b_s; \end{matrix} q, x\right] $$

$$ - q^{a+r} x \frac{(1-q^{b-a+1})}{(1-q^{c+r})} \sum_{i=0}^{s-2} \left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] $$

$$ \times\ {}_{s+1}F_s\left[\begin{matrix} q^{a+r+1}, & q^{b+r}, & a_3 q, a_4 q, \dots\dots, a_s q, a_{s+1} q; \\ & & q^{c+r+1}, b_2 q, b_3 q, \dots\dots, b_s q; \end{matrix} q, x\right] \qquad \dots(3.12) $$

where Re(b_{s-i}) > 0, for all i = 0, 1, · · · , s− 2 and |x| < 1.

We conclude with the remark that the results deduced in the present paper appears to be a new contribution to the theory of generalized hypergeometric series. Secondly, one can easily obtain number of recurrence relations for the generalized hypergeometric functions by the applications of iterated q-integral representation for ${}_{s+1}F_S(\cdot)$.

Remark: If we put r = 0 in all the results, we get the results of [6].

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