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A Fixed Point Theorem In Complete Fuzzy 3-Metric Space Through Rational Expression

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ABSTRACT: Fuzzy metric space have introduced in many ways. We find some fixed point theorem in complete fuzzy 3-metric space through rational expression .Our paper is generalization form of Binayak Choudhary and Krishnapada Das [1] for Fuzzy 3-metric space motivated by Sushil Sharma [10].

I. INTRODUCTION

Fuzzy metric space have been introduced in many ways amongst specially to mention, fuzzy metric spaces were introduced by Kramosil and Michalek [7]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [7] and modified by George and Veeramani [5] to obtain Hausdorff topology for this kind of fuzzy metric space. Recently, Gregori and Sepena (2002) [6] extended Banach fixed point theorem to Fuzzy contraction mappings on complete fuzzy metric space in the sense of George and Veermani [5]. It is remarkable that Sharma, Sharma and Iseki [9] studied for the first time contraction type mappings in 2-metric space. Wenzhi [12] and many others initiated the study of Probabilistic 2-metric spaces. As we know that 2-metric space is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area of function in Euclidean spaces.

Our work demonstrates the fact that other types of contractions are possible in Fuzzy metric space.

II. PRELIMINARIES

Definition 2.1 : (Kramosil and Michalek 1975) A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if it satisfies the following conditions :

(i) $*(1,a) = a$, $*(0,0) = 0$
(ii) $*(a, b) = *(b, a)$
(iii) $*(c, d) \geq *(a, b)$ whenever $c \geq a$ and $d \geq b$
(iv) $*(*(a, b), c) = *(a, *(b, c))$ where a, b, c, $d\in [0,1]$

Definition 2.2 : (Kramosil and Michalek 1975) The 3-tuple (X,M, $*$) is said to be a fuzzy metric space if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$
(ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y) : [0,\infty[\rightarrow[0,1]$ is left-continuous, where $x, y, z \in X$ and $t, s > 0$.

In order to introduced a Hausdroff topology on the fuzzy metric space, in (Kramosil and Michalek 1975) the following definition was introduced.

Definition 2.3 : (George and Veermani 1994) The 3-tuple (X,M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times]0,\infty[$ satisfying the following conditions :

(i) $M(x, y, t) > 0$
(ii) $M(x, y, t) = 1$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) :]0,\infty[\rightarrow[0,1]$ is continuous,

where $x, y, z \in X$ and $t, s > 0$.

Definition 2.4 : (George and Veermani 1994) In a metric space (X,d) the 3-tuple (X, Md,*) where $Md(x, y, t) = t / (t + d(x, y))$ and $a*b = ab$ is a fuzzy metric space . This Md is called the standard fuzzy metric space induced by d.

Definition 2.5 : A binary operation $* : [0,1] \times [0,1] \times [0,1]\rightarrow[0,1]$ is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition 2.6 : The 3-tuple $(X,M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

(FM'-1) $M(x, y, z, 0) = 0$,

(FM'-2) $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,

(FM'-3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$, (Symmetry about three variables)

(FM'-4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1)*M(x, u, z, t_2)* M(u, y, z, t_3)$

(This corresponds to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

(FM'-5) $M(x, y, z, .) : [0,1)\to[0,1]$ is left continuous.

Definition 2.7 : Let $(X, M, *)$ is a fuzzy 2-metric space :

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$, if

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1$$

for all $a\in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$$

for all $a\in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.8 : A function M is continuous in fuzzy 2-metric space iff whenever $x_n\to x$, $y_n\to y$, then

$$\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a,t)$$

for all $a\in X$ and $t > 0$.

Definition 2.9 : Two mappings A and S on fuzzy 2-metric space X are weakly commuting iff

$M(ASu, SAu, a, t) \geq M(Au, Su, a, t)$ for all $u, a \in X$ and $t > 0$.

Definition 2.10 : A binary operation $* : [0,1]^4 \to[0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1* b_1* c_1* d_1\leq a_2* b_2* c_2 * d_2$ whenever $a_1\leq a_2$, $b_1\leq b_2$, $c_1\leq c_2$ and $d_1\leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in [0,1].

Definition 2.11 : The 3-tuple $(X, M, *)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0,\infty)$ satisfying the following conditions : for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$.

(FM''-1) $M(x, y, z, w, 0) = 0$,

(FM''-2) $M(x, y, z, w, t) = 1$ for all $t > 0$, (only when the three simplex $\langle x, y, z, w \rangle$ degenerate)

(FM''-3) $M(x, y, z, w, t) =M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) = \ldots$

(FM''-4) $M(x, y, z, w, t_1 +t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1)*M(x, y, u, w, t_2)*M(x, u, z, w, t_3)*M(u, y, z, w, t_4)$

(FM''-5) $M(x, y, z, w, .) : [0,1)\to[0,1]$ is left continuous.

Definition 2.12 : Let $(X, M, *)$ be a fuzzy 3-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$, if

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$$

for all $a, b \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$$

for all $a, b \in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.13 : A function M is continuous in fuzzy 3-metric space iff whenever $x_n\to x$, $y_n\to y$

$$\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$$

for all $a, b \in X$ and $t > 0$.

Definition 2.14 : Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff

$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, b, t)$ for all $u, a, b \in X$ and $t > 0$.

Remark : Definitions and prepositions from Gregori and Sepena 2002 [6] , Kumar and Chugh 2001 [8] are also used to prove our theorem.

III. MAIN RESULT

Theorem : Let (X,M, *) be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy and T , R and S be mappings from (X ,M ,*) into itself satisfying the following conditions :

$$T(X) \subseteq R(X) \text{ and } T(X) \subseteq S(X)$$

$$\frac{1}{M(T(x),T(y),a,b,t)} - 1 \leq k\left(\frac{1}{L(x,y,a,b,t)} - 1\right)$$

with 0 < k < 1 and

$$L(x, y, a, b, t) = \min\left\{\begin{array}{c} M(Rx,Sy,a,b,t), M(Sx,Ry,a,b,t), M(Rx,Tx,a,b,t), \\ M(Ry,Ty,a,b,t), M(Sx,Tx,a,b,t), M(Sy,Ty,a,b,t), \\ \frac{M(Sx,Ry,a,b,t)M(Rx,Tx,a,b,t)}{M(Rx,Sy,a,b,t)}, \frac{M(Sx,Tx,a,b,t)M(Sy,Ty,a,b,t)}{M(Ry,Ty,a,b,t)} \end{array}\right\}$$

The pairs T, S and T , R are compatible. R, T and S are w-continuous.
Then R, T and S have a unique common fixed point.

Proof : Let $x_0 \in X$ be an arbitrary point of X. Since $T(X) \subseteq R(X)$ and $T(X) \subseteq S(X)$, we can construct a sequence $\{x_n\}$ in X such that

$$Tx_{n-1} = Rx_n = Sx_n$$

Now,

$$L(x_n, x_{n+1}, a, b, t) = \min\left\{\begin{array}{c} M(Rx_n,Sx_{n+1}a,b,t), M(Sx_n,Rx_{n+1},a,b,t), M(Rx_n,Tx_n,a,b,t), \\ M(Rx_{n+1},Tx_{n+1},a,b,t), M(Sx_n,Tx_n,a,b,t), M(Sx_{n+1},Tx_{n+1},a,b,t), \\ \frac{M(Sx_n,Rx_{n+1},a,b,t)M(Rx_n,Tx_n,a,b,t)}{M(Rx_n,Sx_{n+1}a,b,t)}, \frac{M(Sx_n,Tx_n,a,b,t)M(Sx_{n+1},Tx_{n+1},a,b,t)}{M(Rx_{n+1},Tx_{n+1},a,b,t)} \end{array}\right\}$$

$$= \min\left\{\begin{array}{c} M(Tx_{n-1},Tx_n,a,b,t), M(Tx_{n-1},Tx_n,a,b,t), M(Tx_{n+1},Tx_n,a,b,t), \\ M(Tx_n,Tx_{n+1},a,b,t), M(Tx_{n-1},Tx_n,a,b,t), M(Tx_n,Tx_{n+1},a,b,t), \\ \frac{M(Tx_{n-1},Tx_n,a,b,t)M(Tx_{n+1},Tx_n,a,b,t)}{M(Tx_{n-1},Tx_n,t)}, \frac{M(Tx_{n-1},Tx_n,a,b,t)M(Tx_n,Tx_{n+1},a,b,t)}{M(Tx_n,Tx_{n+1},a,b,t)} \end{array}\right\}$$

$$= \min\{M(Tx_{n-1},Tx_n,a,b,t), M(Tx_n,Tx_{n+1},a,b,t)\}$$

We now claim that $M(Tx_{n-1},Tx_n,a,b,t) < M(Tx_n,Tx_{n+1},a,b,t)$

Otherwise we claim that $M(Tx_{n-1},Tx_n,a,b,t) \geq M(Tx_n,Tx_{n+1}a,b,t)$

i.e. $L(x_n, x_{n+1}, a, b, t) = M(Tx_n,Tx_{n+1}, a, b, t)$

$\therefore$ $$\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1\right)$$

which is a contradiction.

Hence, $$\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1\right)$$

$\therefore$ $\{Tx_n\}$ is a fuzzy contractive sequence in (X,M,*). So $\{Tx_n\}$ is a Cauchy sequence in (X,M,*).
As X is a complete fuzzy metric space, $\{Tx_{n-1}\}$ is convergent . So, $\{Tx_{n-1}\}$ converges to some point z in X.
$\therefore$ $\{Tx_{n-1}\}$, $\{Rx_n\}$, $\{Sx_n\}$ converges to z. By w-continuity of R, S and T, there exists a point u in X such that $x_n \to u$ as $n\to\infty$ and so $\ln Rx_n = \ln Sx_n = \ln Tx_{n-1} = z$ implies

$$Ru = Su = Tu = z$$

Also by compatibility of pairs T, S and T, R and Tu = Ru = Su = z implies
Tz = TRu = RTu = Rz and Tz = TSu = STu = Sz

Therefore, Tz = Rz = Sz

We now claim that Tz = z.

If not $\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1\right)$

$$L(z,u,a,b,t) = \min\left\{\begin{matrix} M(Rz,Su,a,b,t), M(Sz,Ru,a,b,t), M(Rz,Tz,a,b,t), \\ M(Ru,Tu,a,b,t), M(Sz,Tz,a,b,t), M(Su,Tu,a,b,t), \\ \frac{M(Sz,Ru,a,b,t)M(Rz,Tz,a,b,t)}{M(Rz,Su,a,b,t)}, \frac{M(Sz,Tz,a,b,t)M(Su,Tu,a,b,t)}{M(Ru,Tu,a,b,t)} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} M(Tz,z,a,b,t), M(Tz,z,a,b,t), M(Tz,Tz,a,b,t), \\ M(z,z,a,b,t), M(Tz,Tz,a,b,t), M(z,z,a,b,t), \\ \frac{M(Tz,z,a,b,t)M(Tz,Tz,a,b,t)}{M(Tz,z,a,b,t)}, \frac{M(Tz,Tz,a,b,t)M(z,z,a,b,t)}{M(z,z,a,b,t)} \end{matrix}\right\}$$

$$= \min\{M(Tz,z,a,b,t), M(Tz,z,a,b,t), 1,1,1,1,1,1\}$$

$$= M(Tz,z,a,b,t)$$

$\therefore$ $\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1\right)$

which is a contradiction.

Hence Tz = z

So z is a common fixed point of R, T and S.

Now suppose v ≠ z be another fixed point of R, T and

$\therefore$ $\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1\right)$

$$L(v,u,a,b,t) = \min\left\{\begin{matrix} M(Rv,Sz,a,b,t), M(Sv,Rz,a,b,t), M(Rv,Tv,a,b,t), \\ M(Rz,Tz,a,b,t), M(Sv,Tv,a,b,t), M(Sz,Tz,a,b,t), \\ \frac{M(Sv,Rz,a,b,t)M(Rv,Tv,a,b,t)}{M(Rv,Sz,a,b,t)}, \frac{M(Sv,Tv,a,b,t)M(Sz,Tz,a,b,t)}{M(Rz,Tz,a,b,t)} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} M(v,z,a,b,t), M(v,z,a,b,t), M(v,v,a,b,t), \\ M(z,z,a,b,t), M(v,v,a,b,t), M(z,z,a,b,t), \\ \frac{M(v,z,a,b,t)M(v,v,a,b,t)}{M(v,z,a,b,t)}, \frac{M(v,v,a,b,t)M(z,z,a,b,t)}{M(z,z,a,b,t)} \end{matrix}\right\}$$

$$= \min\{M(v,z,a,b,t), M(v,z,a,b,t), 1,1,1,1,1,1\}$$

$$= M(v,z,a,b,t)$$

$\therefore$ $\frac{1}{M(Tx_n,Tx_{n+1},a,b,t)} - 1 \leq k\left(\frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1\right)$

which is a contradiction. Hence v = z.

Thus R, T and S have a unique common fixed point.

This completes the proof.

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