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Determination of Lyapunov Exponents in Discrete Chaotic Models

Dr. Nabajyoti Das
Department of Mathematics,
J.N. College Kamrup, Assam, India

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ABSTRACT: This paper discusses the Lyapunov exponents as a quantifier of chaos with two dimensional discrete chaotic model:

$$\mathbf{F(x,y) = (y,\ \mu x+\lambda y - y^3\ ),}$$

Where, μ and λ are adjustable parameters. Our prime objective is to find First Lyapunov exponent, Second Lyapunov exponent and Maximal Lyapunov exponent as the notion of exponential divergence of nearby trajectories indicating the existence of chaos in our concerned map. Moreover, these results have paused many challenging open problems in our field of research.

Key Words: Discrete System/ Lyapunov Exponent / Quantifier of Chaos
2010 AMS Subject Classification : 37 G 15, 37 G 35, 37 C 45

I. INTRODUCTION

Distinguishing deterministic chaos from noise has become an important problem in many diverse fields . This is due, in part, to the availability of numerical algorithms for quantifying chaos using experimental time series. In particular, methods exist for calculating correlation dimension (D_2) Kolmogorov entropy, and Lyapunov exponents. Dimension gives an estimate of the complexity of the system ; and entropy and Lyapunov exponents give an estimate of the level of chaos in the dynamical system, [1,2,7] .

In mathematics Lyapunov functions are functions which can be used to prove the stability of a certain fixed point (or a periodic point) in a dynamical system or autonomous differential equation. Named after the Russian mathematician Aleksandr Mikhailovich Lyapunov, Lyapunov functions are important to stability theory and control theory.

The rate of separation can be different for different orientations of initial separation vector. Thus, there is a whole spectrum of Lyapunov exponents—the number of them is equal to the number of dimensions of the *phase space*. It is common to just refer to the largest one, i.e. to the Maximal Lyapunov exponent (MLE), because it determines the predictability of a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic. Note that an arbitrary initial separation vector will typically contain some component in the direction associated with the MLE, and because of the exponential growth rate the effect of the other exponents will be obliterated over time. For a dynamical system with evolution equation f^t in a n–dimensional phase space, the spectrum of Lyapunov exponents in general, depends on the starting point x_0. However, we are usually interested in the attractor (or attractors) of a dynamical system, and there will normally be one set of exponents associated with each attractor. The choice of starting point may determine on which attractor the system ends up on, if there is more than one. The Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix, [3,5,8].

The J^t matrix describes how a small change at the point x_0 propagates to the final point $f^t(x_0)$. The limit defines a matrix $L(x_0)$ (the conditions for the existence of the limit are given by the Oseledec theorem).

Lyapunov exponents may provide a more useful characterization of chaotic systems.

For time series produced by dynamical systems, the presence of a positive characteristic exponent indicates chaos. Furthermore, in many applications it is sufficient to calculate only the largest Lyapunov exponent.

II. MATEIAL AND METHODS

Lyapunov Exponent, [3, 4, 6, 7] :

Lyapunov exponent is calculated by eigen values of the limit of the following expression: $(J_0.J_1.............J_n)^{1/n}$ where n tends to infinity, and J_i is the Jacobian of the function at the iterated point (x_i , y_i). For the evaluation of Lyapunov exponent, we have taken an initial point and iterated it say for two thousand time so that we are sufficiently closer to the fixed point. We find $(J_0.J_1.............J_n)$, where n =2000 say and calculate the Eigen values of that resultant matrix. Then log (eigen value) /n is the Lyapunov exponent. To confirm the existence of chaos, the following are carried out.

(i) First Lyapunov Exponent and its Analysis
(ii) Second Lyapunov Exponent and its Analysis
(iii) Maximal Lyapunov Exponent and its Analysis

Let us take an initial point (x_0 , y_0) for the model $F(x,y) = (y,\ \mu x + \lambda y - y^3)$. We now calculate the Jacobian of F , say J_0 as

$J_0 = \begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_0}^2 \end{pmatrix}$. We then find the Jacobian of F^2 (second iteration of F) as

$J_1 = \begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_0}^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_1}^2 \end{pmatrix}$, where $x_1 = y_0$ and $y_1 = \mu x_0 + \lambda y_0 - {y_0}^3$.

Similarly, we can calculate the Jacobian J_2 of F^3 as

$J_2 = \begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_0}^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_1}^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_2}^2 \end{pmatrix}$, where $x_2 = y_1$ and

$$y_2 = \mu x_1 + \lambda y_1 - {y_1}^3$$

Continuing in this way we can determine the Jacobian of F^{n+1} as

$J_n = \begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_0}^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_1}^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_2}^2 \end{pmatrix}\cdots\cdots\cdots\cdots\begin{pmatrix} 0 & 1 \\ \mu & \lambda - 3{y_n}^2 \end{pmatrix}$ and so on.

We next calculate the eigenvalues of J_n .Since it is a 2X2 matrix, it will give us two eigenvalues say , A_1 and A_2 . The two Lyapunov exponents,

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \log A_i \quad \text{for I = 1,2.}$$

If atleast one Lyapunov exponent is positive for some control parameter value then the system is chaotic at that control parameter. However, if the system is dissipative then $\lambda_1 + \lambda_2$ will be negative.

III. MAIN RESULTS

1. First Lyapunov Exponent: Let us consider the parameter- range as (2.5, 3.2 6). Then First Lyapunov Exponent at this parameter range 2.5-3.26 using 2000 iterations are as follows [by employing MATHEMATICA Computer package]:

Table 1: [for First Lypunov Exponent].

{2.500000,-0.346915},{2.510000,-0.346901},{2.520000,-0.346578},
{2.530000,-0.346659},{2.540000,-0.346932},{2.550000,-0.347309},
{2.560000,-0.347426},{2.570000,-0.347075},{2.580000,-0.346728},
{2.590000,-0.346574},{2.600000,-0.346862},{2.610000,-0.346788},
{2.620000,-0.346736},{2.630000,-0.346618},{2.640000,-0.347125},
{2.650000,-0.347724},{2.660000,-0.347576},{2.670000,-0.347253},
{2.680000,-0.346628},{2.690000,-0.346798},{2.700000,-0.346616},
{2.710000,-0.347050},{2.720000,-0.346575},{2.730000,-0.346737},
{2.740000,-0.346771},{2.750000,-0.044131},{2.760000, 0.190023}, {2.770000,0.092523},{2.780000,-0.346696},{2.790000,-0.346574},
{2.800000,-0.346617},{2.810000,-0.346914},{2.820000,-0.346963},
{2.830000,-0.346617},{2.840000,-0.346648},{2.850000,-0.346636},
{2.860000,-0.346748},{2.870000,-0.348341},{2.880000,-0.346708},
{2.890000,-0.346769},{2.900000,-0.346764},{2.910000,-0.346675},
{2.920000,-0.346762},{2.930000,-0.346585},{2.940000,-0.347111},
{2.950000,-0.347072},{2.960000,-0.256489},{2.970000,-0.173527},
{2.980000,-0.110796},{2.990000,-0.060673},{3.000000,-0.019065},
{3.010000,-0.110663},{3.020000,-0.278214},{3.030000,-0.346447},
{3.040000,-0.346606},{3.050000,-0.346495},{3.060000,-0.346395},
{3.070000,-0.346741},{3.080000,-0.346273},{3.090000,-0.346517},
{3.100000,-0.347734},{3.110000,-0.348122},{3.120000,-0.346496},
{3.130000,-0.346265},{3.140000,-0.346590},{3.150000,-0.346082},
{3.160000,-0.346016},{3.170000,-0.346431},{3.180000,-0.345450},
{3.190000,-0.140837},{3.200000,-0.032816}

We plot the above points in order to get the graph of First Lyapunov Exponent as elicited below:

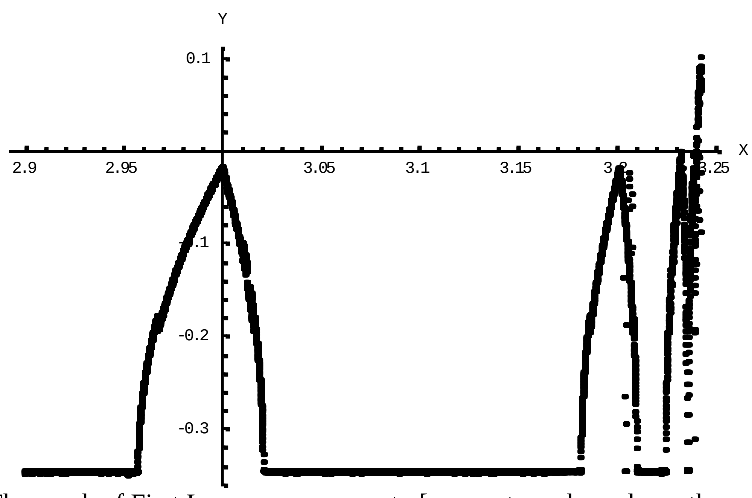

Fig 1. The graph of First Lyapunov exponent: [parameter values along the x- axis and the corresponding values of the first Lyapunov exponents along the y – axis.

Remark: Negative values indicate stability and positive values indicate chaos .

The figure shows that when the parameter values are greater than 3.25 (approx), positive Lyapunov exponents are observed and thus chaos exists in our system.

2. Second Lyapunov Exponent :

Let us consider the parameter- range as (2.5, 3.26). Then Second Lyapunov Exponents at this parameter range 2.5-3.26 using 2000 iterations are as follows [first coordinate represents a parameter value and the second coordinate represents corresponding second Lyapunov exponent].

Table 2: [for Second Lyapunov Exponent]

{2.500000,-0.346915},{2.510000,-0.346901},{2.520000,-0.346578},
{2.530000,-0.346659},{2.540000,-0.346932},{2.550000,-0.347309},
{2.560000,-0.347426},{2.570000,-0.347075},{2.580000,-0.346728},
{2.590000,-0.346574},{2.600000,-0.346862},{2.610000,-0.346788},
{2.620000,-0.346736},{2.630000,-0.346618},{2.640000,-0.347125},
{2.650000,-0.347724},{2.660000,-0.347576},{2.670000,-0.347253},
{2.680000,-0.346628},{2.690000,-0.346798},{2.700000,-0.346616},
{2.710000,-0.347050},{2.720000,-0.346575},{2.730000,-0.346737},
{2.740000,-0.346771},{2.750000,-0.026255},{2.760000, -0.190023}, {2.770000,0.110854},{2.780000,-0.346696},{2.790000,-0.346574},
{2.800000,-0.346617},{2.810000,-0.346914},{2.820000,-0.346963},
{2.830000,-0.346617},{2.840000,-0.346648},{2.850000,-0.346636},
{2.860000,-0.346748},{2.870000,-0.348341},{2.880000,-0.346708},
{2.890000,-0.346769},{2.900000,-0.346764},{2.910000,-0.346675},
{2.920000,-0.346762},{2.930000,-0.346585},{2.940000,-0.347111},
{2.950000,-0.347072},{2.960000,-0.256489},{2.970000,-0.155898},
{2.980000,-0.092791},{2.990000,-0.042726},{3.000000,-0.000003},
{3.010000,-0.096770},{3.020000,-0.278214},{3.030000,-0.346447},
{3.040000,-0.346606},{3.050000,-0.346495},{3.060000,-0.346395},
{3.070000,-0.346741},{3.080000,-0.346273},{3.090000,-0.346517},
{3.100000,-0.347734},{3.110000,-0.348122},{3.120000,-0.346496},
{3.130000,-0.346265},{3.140000,-0.346590},{3.150000,-0.346082},
{3.160000,-0.346016},{3.170000,-0.346431},{3.180000,-0.345450},
{3.190000,-0.121727},{3.200000,-0.013244},

If we plot the above points, we obtain the following graph of second Lyapunov exponent

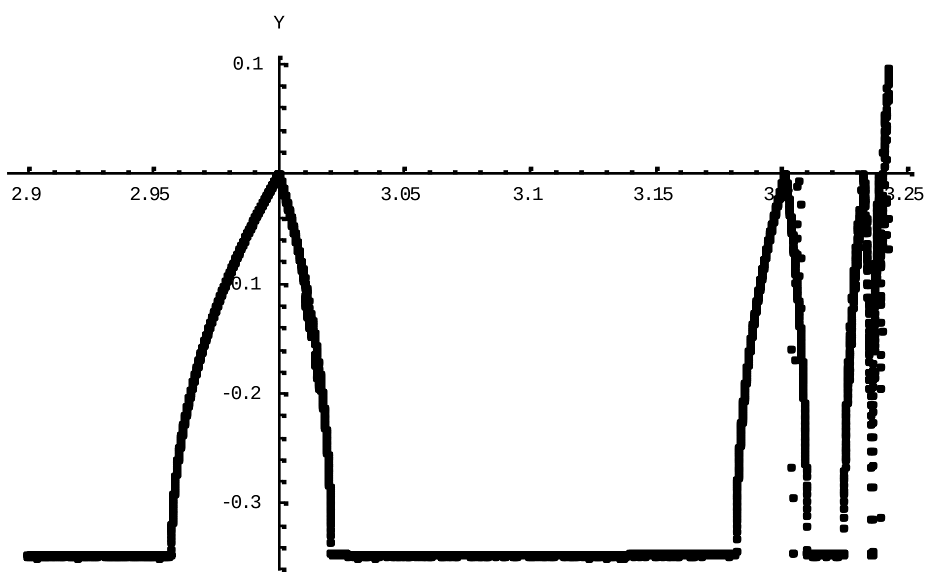

Fig 2 : Graph of second Lyapunov exponent (parameter values along the x- axis and the corresponding second Lyapunov exponent along the y- axis).

Remark: The graph shows that when the parameter values are greater than 3.25 (approx.), there are positive Lyapunov exponents confirming the existence of chaos.

3. Maximum Lyapunov Exponent : Let us consider the parameter- range as (2.5 , 3.2). Then Maximum Lyapunov Exponent at this parameter range 2.5-3.2 using 2000 iterations are as follows.

Table 3: [for Maximum Lyapunov Exponent].

{2.50 , -0.34691508406819}, {2.510 ,-0.34690088756770},
{2.520,-0.34657795786979}, {2.530 ,-0.34665860880707},
{2.540 ,-0.34693224584640}, {2.550 ,-0.34730852188329},
{2.560 ,-0.34742571527168}, {2.570,-0.34707502252587},
{2.5800 ,-0.34672845683957}, {2.590 ,-0.34657380249449},
{2.60 ,-0.34686199132869}, {2.610 ,-0.34678774794881},
{2.620 ,-0.34673638174957}, {2.630 ,-0.34661760511891},
{2.640 ,-0.34712487345509}, {2.650 ,-0.34772370453388},
{2.660 ,-0.34757607461325}, {2.6700 ,-0.34725308752558},
{2.680 ,-0.34662839297866}, {2.690 ,-0.34679777561419},
{2.70 ,-0.34661595700258}, {2.710 ,-0.34705018441629},
{2.720 ,-0.34657494045799}, {2.730,-0.34673697448748},
{2.73999999999999,-0.34677138633359},{2.74999999999999,-0.02625469913338},
{2.75999999999999,-0.19002315837202}, {2.76999999999999,0.11085431594895},
{2.77999999999999,-0.34669584579425},{2.78999999999999,-0.34657379446624},
{2.79999999999999,-0.34661653211568},{2.80999999999999,-0.34691393256321},
{2.81999999999999,-0.34696318777213},{2.82999999999999,-0.34661699456539},
{2.83999999999999,-0.34664785919627},{2.84999999999999,-0.34663611469232},
{2.85999999999999,-0.34674757759330},{2.86999999999999,-0.34834059260088},
{2.87999999999999,-0.34670770039304}, {2.88999999999999,-0.34676911095608},
{2.89999999999999,-0.34676414801158}, {2.90999999999999,-0.34667469499401},
{2.91999999999999,-0.34676188298241}, {2.92999999999999,-0.34658514587783},
{2.93999999999999,-0.34711059747737}, {2.94999999999999,-0.34707157380425},
{2.95999999999999,-0.25648944177257}, {2.96999999999999,-0.15589783894841},
{2.97999999999999,-0.09279142591047}, {2.98999999999999,-0.04272563624209}, {2.99999999999999,-
0.00000279745808},{3.00999999999999,-0.09677045581330}, {3.01999999999999,-
0.27821378894670},{3.02999999999999,-0.34644702822118}, {3.03999999999999,-
0.34660635490615},{3.04999999999999,-0.34649487807087}, {3.05999999999999,-
0.34639516937635},{3.06999999999999,-0.34674146667689},
{3.07999999999999,-0.34627257047845},{3.08999999999999,-0.34651685663619}, {3.09999999999999,-
0.34773396323411},{3.10999999999999,-0.34812192822421}, {3.11999999999999,-
0.34649595856214},{3.12999999999999,-0.34626472692972}, {3.13999999999999,-
0.34659027799119},{3.14999999999999,-0.34608209183004}, {3.15999999999999,-
0.34601594434633},{3.16999999999999,-0.34643137101169}, {3.17999999999999,-
0.34545023892224},{3.18999999999999,-0.12172680968631}, {3.19999999999999,-0.01324379044788} .

If we plot the above points, we obtain the following graph of maximum Lyapunov exponents.

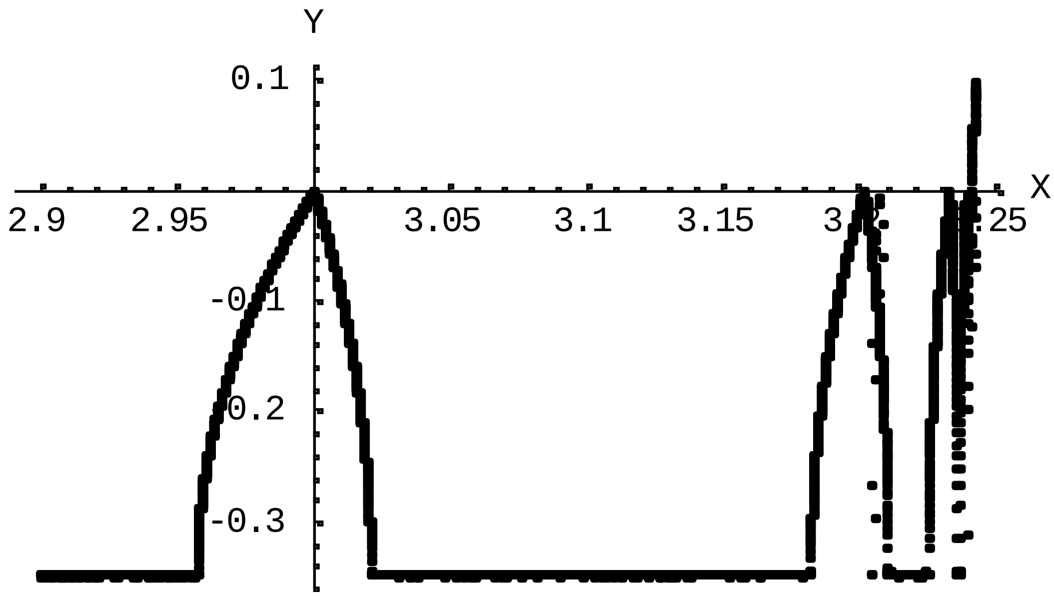

Fig 3. Graph of maximum Lyapunov exponent. [Parameter values along the x- axis and the corresponding Maximum Lyapunov exponent along the y- axis].

Remark: The graph shows that when the parameter values are greater than 3.25 (approx.), there are positive Lyapunov exponents confirming the existence of chaos .

Open Problems: (i) Can we extend these results to three dimensional or higher dimensional models ?
(ii) Can we have some models where first and second Lyapunov exponents do not indicate the existence of chaos, and only **maximum Lyapunov exponent** indicates the existence of chaos ?

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