International Journal of Theoretical and Applied Science **3**(2): 107-115(2011) **ISSN No. (Print) : 0975-1718**
ISSN No. (Online) : 2249-3247

Some Fixed Point and Common Fixed Point Theorems in Banach Spaces for Rational Expression

Rajesh Shrivastava*, Jitendra Singhvi*, Ramakant Bhardwaj** and Shyam Patkar**

**Department of Mathematics, Government Science and Commerce College, Benazeer, Bhopal, (MP)*
***Department of Mathematics, Truba Institute of Engineering and I.T., Bhopal, (MP)*

(Received 15 September, 2011, Accepted 20 October, 2011)

ABSTRACT : In the present paper we prove some fixed point and common fixed point theorems in Banach Spaces for new rational expression, which generalize the well known results.

Keywords: AMS: 47H10, 54H25.

I. INTRODUCTION

Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwder [4] and Ghode [6] have independently proved a fixed point theorem for non-expansive mapping.

Many other mathematicians viz; Datson [5] Goebel [6], Goebel and Zlotkienwicz [8], Goebel, Kirk and Simi [9], Iseki [11], Singh and Chatterjee [22], Sharma and Rajput [21], Rajput and Naroliya [20] Pathak and Maity [18], Qureshi and Singh [19], Sharma and Bhagwan [23], Ahmad and Shakil [1], Shahzad and Udomene [24] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [15, 16 and 17] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings. On the same way we establish some fixed point and common fixed point theorems for non-contraction type mappings for two mappings.

II. MAIN RESULTS

Theorem 1.3: Let T be a mapping of a Banach spaces into itself. If T satisfies the following conditions:

$$T^2 = I, \text{ where } I \text{ is identity mapping} \qquad \ldots [1.3(a)]$$

$$\|Tx-Ty\| \le \alpha \frac{\|x-Tx\|\,\|y-Ty\|\,\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2} + \beta \frac{\|y-Ty\|\,\|y-Tx\|\,\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2}$$
$$+\gamma\left[\|x-Tx\|+\|y-Ty\|\right]$$
$$+\delta\left[\|x-Ty\|+\|y-Tx\|\right]+\eta\|x-y\| \qquad \ldots [1.3(b)]$$

$x \ne y$, $with 10\alpha+9\beta+8\gamma+5\delta+\eta<4$ then unique fixed point.

Proof : Suppose x is any point in Banach space X.

Taking $y=\frac{1}{2}(T+I)x,\ z=T(y)$

$$\|z-x\|=\|Ty-T^2x\|=\|Ty-T(Tx)\|$$

$$\|Ty-T(Tx)\| \le \alpha \frac{\|y-Ty\|\,\|Tx-x\|\,\|y-x\|+\|y-Tx\|^3}{\|y-Tx\|^2}$$
$$+\beta \frac{\|Tx-x\|\,\|Tx-Ty\|\,\|y-x\|+\|y-Tx\|^3}{\|y-Tx^2\|}$$
$$+\gamma\left[\|y-Ty\|+\|Tx-x\|\right]+\delta\left[\|y-x\|+\|Tx-Ty\|\right]+\eta\|y-Tx\|\|$$

$$\le \alpha \frac{\|y-Ty\|\,\|Tx-x\|\frac{1}{2}\|x-Tx\|+\frac{1}{8}\|x-Tx\|^3}{\frac{1}{4}\|x-Tx\|^2}$$
$$+\beta \frac{\|Tx-x\|\left[\|Tx-y\|+\|y-Ty\|\right]\frac{1}{2}\|x-Tx\|+\|y-Tx\|^3}{\frac{1}{4}\|x-Tx\|^2}$$
$$+\gamma\left[\|y-Ty\|+\|Tx-x\|\right]+\delta\left[\frac{1}{2}\|x-Tx\|+\|Tx-y\|+\|y-Ty\|\right]+\eta\frac{1}{2}\|x-Tx\|$$

$$= \alpha \frac{\left\{4\|y-Ty\|\,\|Tx-x\|^2+\|x-Tx\|^3\right\}/8}{\frac{1}{4}\|x-Tx\|^2}$$

$$+\beta \frac{\|Tx-x\|\left[\frac{1}{2}\|x-Tx\|+\|y-Ty\|\right]\frac{1}{2}\|x-Tx\|+\frac{1}{8}\|x-Tx\|^3}{\frac{1}{4}\|x-Tx\|^2}$$

$$+\gamma\left[\|y-Ty\|+\|Tx-x\|\right]$$

$$+\delta\left[\frac{1}{2}\|x-Tx\|+\frac{1}{2}\|x-Tx\|+\|y-Ty\|\right]+\eta\frac{1}{2}\|y-Tx\|$$

$$\le \frac{\alpha}{2}\left\{4\|y-Ty\|+\|x-Tx\|\right\}$$

$$+\frac{\beta}{2}\left\{4\|Tx-x\|^2\left[\frac{1}{2}\|x-Tx\|+\|y-Ty\|\right]+\frac{\|x-Tx\|^3}{\|Tx-x\|^2}\right\}$$

$$+\gamma\left[\|y-Ty\|+\|Tx-x\|\right]+\delta\left[\|x-Tx\|+\|y-Ty\|\right]+\frac{\eta}{2}\|x-Tx\|$$

$$=\frac{\alpha}{2}\left[\|x-Tx\|+4\|y-Ty\|\right]+\frac{\beta}{2}\left[2\|x-Tx\|+4\|y-Ty\|\right]$$

$$+\gamma\left[\|y-Ty\|+\|Tx-x\|\right]+\delta\left[\|x-Tx\|+\|y-Ty\|\right]+\frac{\eta}{2}\|x-Tx\|$$

$$=\|x-Tx\|\left[\frac{\alpha}{2}+\beta+\gamma+\delta+\frac{\eta}{2}\right]+\|y-Ty\|\left[2\alpha+2\beta+\gamma+\delta\right] \quad \text{... (1.3 } x\text{)}$$

$$\|z-x\|\le\frac{1}{2}\left[\alpha+2\beta+2\gamma+2\delta+\eta\right]+\|y-Ty\|\left[2\alpha+2\beta+\gamma+\delta\right]$$

Now for $\|u-x\|$

$$\|u-x\|=\|2y-z-x\|=\|Tx-Ty\|$$

$$\le \alpha\frac{\|x-Tx\|\,\|y-Ty\|\,\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2}$$

$$+\beta\frac{\|y-Ty\|\,\|y-Tx\|\,\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2}$$

$$+\gamma\left[\|x-Tx\|+\|y-Ty\|\right]+\delta\left[\|x-Ty\|+\|y-Tx\|\right]+\eta\|x-y\|$$

$$\|u-x\|\le\alpha\frac{\frac{1}{2}\|x-T\|^2\|y-Ty\|+\frac{1}{8}\|x-Tx\|^3}{\frac{1}{4}\|x-Tx\|^2}$$

$$+\beta\frac{\|y-Ty\|\frac{1}{2}\|x-Tx\|\left[\frac{1}{2}\|Tx-x\|\right]+\|x-y\|^3}{\frac{1}{4}\|x-Tx\|^2}$$

$$+\gamma\left[\|x-Tx\|+\|y-Ty\|\right]+\delta\left[\frac{1}{2}\|x-Tx\|+\frac{1}{2}\|x-Tx\|\right]+\frac{\eta}{2}\|x-Tx\|$$

$$=\alpha\frac{\left[4\|x-Tx\|^2\|y-Ty\|+\|x-Tx\|^3\right]}{2\|x-Tx\|^2}$$

$$+\beta\frac{\left[\frac{1}{2}\|y-Ty\|\,\|x-Tx\|\left[\frac{1}{2}\|x-Tx\|\right]+\frac{1}{8}\|x-Tx\|^3\right]}{\frac{1}{4}\|x-Tx\|^2}$$

$$+\gamma\left[\|x-Tx\|+\|y-Ty\|\right]+\delta\|x-Tx\|+\frac{\eta}{2}\|x-Tx\|$$

$$=\frac{\alpha}{2}\left[4\|y-Ty\|+\|x-Tx\|\right]+\beta\frac{\left[2\|y-Ty\|\|\|x-Tx\|^2+\|x-Tx\|^3\right]}{2\|x-Tx\|^2}+\gamma\left[\|x-Tx\|+\|y-Ty\|\right]$$

$$+\delta\left[\|x-Tx\|\right]+\frac{\eta}{2}\left[\|x-Tx\|\right]$$

$$=\|x-Tx\|\left[\frac{\alpha}{2}+\frac{\beta}{2}+\gamma+\delta+\frac{\eta}{2}\right]+\|y-Ty\|\left[2\alpha+\beta+\gamma\right]$$

$$+\|x-Tx\|\frac{1}{2}\left[\alpha+\beta+2\gamma+\delta\right]+\|y-Ty\|\left[2\alpha+\beta+\gamma\right] \quad \text{... (13.}\delta\text{)}$$

Now

$$\|z-u\|\le\|z-x\|+\|x-u\|$$

$$\le\frac{1}{2}\|x-Tx\|\left[\alpha+2\beta+2\gamma+2\delta+\eta\right]+\|y-Ty\|\left[2\alpha+2\beta+\gamma+\delta\right]$$

$$+\frac{1}{2}\left\{\left[\alpha+\beta+2\gamma+\delta\right]\|x-Tx\|\right\}+\|y-Ty\|\left[2\alpha+\beta+\gamma\right]$$

$$\|z-u\|\le\frac{1}{2}\|x-Tx\|\left\{\alpha+2\beta+2\gamma+2\delta+\eta+\alpha+\beta+2\gamma+\delta\right\}$$

$$+\|y-Ty\|\left[2\alpha+2\beta+\gamma+\delta+2\alpha+\beta+\gamma\right]$$

$$=\frac{1}{2}\|x-Tx\|\left\{2\alpha+3\beta+4\gamma+3\delta+\eta\right\}+\|y-Ty\|\left[4\alpha+3\beta+2\gamma+\delta\right]$$

On the other hand

$$\|z-u\|=\|T(y)-(2y-z)\|$$

$$=\|T(y)-2y+T(y)\|$$

$$\|z-u\|=2\|Ty-y\|$$

So

$$2\|Ty-y\|\le\frac{1}{2}\|x-Tx\|\left[2\alpha+3\beta+4\gamma+3\delta+\eta\right]+\|y-Ty\|\left[4\alpha+3\beta+2\gamma+\delta\right]$$

$$\|Ty-y\|\le\frac{\left[\alpha+\frac{3}{2}\beta+2\gamma+\frac{3}{2}\delta+\frac{\eta}{2}\right]\|x-Tx\|}{2-\left[4\alpha+3\beta+2\gamma+\delta\right]}$$

$$\|Ty-y\|\le S\|x-Tx\|$$

where $S=\dfrac{\left[2\alpha+3\beta+4\gamma+3\delta+\eta\right]}{4-\left[8\alpha+6\beta+4\gamma+2\delta\right]}<1$

because $10\alpha+9\beta+8\gamma+5\delta+\eta<4$

Let $R=\frac{1}{2}\left[T+I\right]$, then

$$\| R^2(x) - R(x) \| = \| R(R(x)) - R(x) \|$$
$$= \| R(y) - y \| = \frac{1}{2} \| y - Ty \|$$
$$< \frac{S}{2} \| x - Tx \|$$

By the definition of R we claim that $\{R^n(x)\}$ is a Cauchy sequence in X. By the completeness of X, $\{R^n(x)\}$ is converges to so element x_0, in X. So

$$\lim_{n\to\infty} \{R^n(x)\} = x_0$$

So $\{R(x_0)\} = x_0$, Hence $T(x_0) = x_0$

So x_0 is a fixed point of T.

Uniqueness : If possible let $y_0 \neq x_0$ is another fixed point of T. Then $\| x_0 - y_0 \| = \| Tx_0 - T(y_0) \|$

$$\leq \alpha \| \frac{\| x_0 - Tx_0 \| \| y_0 - Ty_0 \| \| x_0 - Ty_0 \| \| \| x_0 - y_0 \|^3}{\| x_0 - y_0 \|^2}$$
$$+ \beta \| \frac{\| y_0 - Ty \| \| \| y_0 - Tx_0 \| \| x_0 - Ty_0 \| + \| x_0 - y_0 \|^3}{\| x_0 - y_0 \|^2}$$
$$+ \gamma [\| x_0 - Tx_0 \| + \| y_0 - Ty_0 \|] + \delta [\| x_0 - Ty_0 \| + \| y_0 - Tx_0 \|] + \eta \| x_0 - y_0 \|$$
$$= \alpha \| x_0 - y_0 \| + \beta \| x_0 - y_0 \| + 2\delta \| x_0 - y_0 \| + \eta \| x_0 - y_0 \|$$
$$\| x_0 - y_0 \| \leq (\alpha + \beta + 2\delta + \eta) \| x_0 - y_0 \|$$

Which is a contradiction so $x_0 = y_0$

Hence Fixed Point is unique.

Now we prove common fixed point theorem for two mappings.

Theorem 1.4. Let K be closed and convex subset of a Banach space X, let $T : K \to K$ and $G : K \to K$ satisfies the following condition.

(1.4.1) (a) T and G commute

(1.4.2) (b) $T^2 = I$ and $G^2 = I$, where I denotes the identity mapping.

$$(1.4.3)\ (c)\ \| Tx - Ty \| \leq \alpha \| \frac{\| Gx - Tx \| \| Gy - Ty \| \| Gx - Ty \| + \| Gx - Gy \|^3}{\| Gx - Gy \|^2}$$
$$+ \beta \| \frac{\| Gy - Ty \| \| Gy - Tx \| \| \| Gx - Ty \| + \| Gx - Gy \|^3}{\| Gx - Gy \|^2}$$
$$+ \gamma [\| Gx - Tx \| + \| Gy - Ty \|] + \delta [\| Gx - Ty \| + \| Gy - Tx \|] + \eta \| Gx - Gy \|$$

for every $x, y \in X, \alpha, \beta, \gamma, \delta, \eta \in [0,1[$ with $x \neq y$ and $\|Gx - Gy\| \neq 0$ And then there exists a common fixed point of T and G such that $T(x_0) = x_0$ and $G(x_0) = x_0$ with $10\alpha + 9\beta + 8\gamma + 5\delta + n < 4$ then T and G have unique fixed point.

Proof : Suppose x is point in Banach space X. It is clears that $(TG)^2 = I$

Now

$$\| TG.G(x)-TG.G(y)\| \le \alpha \frac{\| G(G^2y)-T(G^2x)\| \, \| G(G^2y)-T(G^2y)\| \, \| G(G^2x)-T(G^2y)\| + \| G(G^2x)-G(G^2y)\|^3}{\| G(G^2x)-G(G^2y)\|^2}$$

$$+\beta \frac{\| G(G^2y)-T(G^2y)\| \; \| G(G^2y)-T(G^2x)\| \; \| G(G^2x)-T(G^2y)\| + \| G(G^2x)-G(G^2y)+\|^3}{\| G(G^2x)-G(G^2y)\|^2}$$

$$+\gamma\left[\| G(G^2x)-T(G^2x)\| + \| G(G^2y)-T(G^2y)\| \right] + \delta\left[\| G(G^2x)-T(G^2y)\| + \| G(G^2y)-T(G^2x)\|\right]$$

$$+\eta\left[\| G(G^2x)-T(G^2y)\|\right]$$

$$=\alpha \| \frac{\| Gx-TG(Gx)\| \, \| G(y)-TG(Gy)\| \| \| G(x)-TG(Gy)\| + \| Gx-Gy\|^3}{\| Gx-Gy\|^2}$$

$$+\beta \frac{\| G(y)-TG(Gy)\| \; \| Gy-TG(Gx)\| \; \| Gx-TG(Gy)\| + \| Gx-Gy\|^3}{\| Gx-Gy\|^2}$$

$$+\gamma\left[\| Gx-TG(Gx)\| + \| G(y)-TG(G(y))\| \right]$$

$$+\delta\left[\| G(x)-TG(G(y))\| + \| G(y)-TG(G(x))\| \right]$$

$$+\eta\left[\| G(x)-G(y)\| \right]$$

Taking $G(x) = p$, $G(y) = q$, where $p \neq q$

$$\| TG.G(x)-TG.G(y)\| = \| TG(p)-TG(q)\|$$

$$\le \alpha \frac{\| p-TG(p)\| \, \| q-TG(q)\| \, \| p-TG(q)\| + \| p-q\|^3}{\| p-q\|^2}$$

$$+\beta \frac{\| q-TG(q)\| \| \| q-TG(p)\| \, \| p-TG(q)\| + \| p-q\|^3}{\| p-q\|^2}$$

$$+\gamma\left[\| p-TG(p)\| + \| q-TG(q)\|\right]$$

$$+\delta\left[\| p-TG(q)\| + \| q-TG(p)\|\right] + \eta \| p-q\|$$

Taking $TG = R$ we get

$$\| R(p)-R(q)\| \le \alpha \frac{\| p-R(p)\| \, \| q-R(q)\| \, \| p-R(q)\| + \| p-q\|^3}{\| p-q\|^2}$$

$$+\beta \| \frac{\| q-R(q)\| \, \| q-R(p)\| \, \| p-R(q)\| + \| p-q\|^3}{\| p-q\|^2}$$

$$+\gamma\left[\| p-R(p)\| + \| q-R(q)\|\right]$$

$$+\delta\left[\| p-R(q)\| + \| q-R(p)\|\right] + \eta \| p-q\|$$

It is clear by theorem 1.3; that $R = TG$ has at least one fixed point say x_0 in K

That is $\quad R(x_0) = TG(x_0) = x_0$ ... (1.4 *d*)

and so $\quad T.(TG)(x_0) = (x_0)$

or $\quad T^2(Gx_0) = T(x_0)$

$G(x_0) = T(x_0)$ ... (1.4 *e*)

Now

$$\| Tx_0 - x_0 \| = \| Tx_0 - T^2(x_0) = \| Tx_0 - T.T(x_0) \|$$
$$\leq \alpha \frac{\| G(x_0) - T(x_0) \| \| GT(x_0) - T(Tx_0) \| \| G(x_0) - T(Tx_0) \| + \| G(x_0) - G(Tx_0) \|^3}{\| G(x_0) - G(Tx_0) \|^2}$$
$$+\beta \frac{\| G(Tx_0) - T(Tx_0) \| \| G(Tx_0) - T(x_0) \| \| G(x_0) - T(Tx_0) \| + \| G(x_0) - G(Tx_0) \|^3}{\| G(x_0) - G(Tx_0) \|^2}$$
$$+\gamma \left[\| G(x_0) - T(x_0) \| + \| G(Tx_0) - T(Tx_0) \| \right]$$
$$+\delta \left[\| G(x_0) - T(Tx_0) \| + \| G(Tx_0) - T(x_0) \| \right]$$
$$+\eta \| G(x_0) - G(Tx_0) \|$$

$\Rightarrow$ $\quad {}_0\| Tx_0 - x_{00} \| \leq [\alpha + \beta + 2\delta + \eta]\, {}_0\| Tx_0 - x_0 \| \quad \because \alpha + \beta + 2\delta + \eta < 1$

So $\quad T(x_0) = x_0$

That is x_0 is the fixed point of T. But $T(x_0) = G(x_0)$ so $G(x_0) = x_0$

Hence x_0 is the fixed point of T and G.

Uniqueness : If possible let $y_0 \neq x_0$ is another common fixed point of T and G.

Then $\| x_0 - y_0 \| = \| T^2(x_0) - T^2(y_0) \| = \| T(T(x_0)) - T(Ty_0) \|$

$$\leq \alpha \frac{\| G(Tx_0) - T(Tx_0) \| \| \| G(Ty_0) - T(Ty_0) \| \| G(Tx_0) - T(Ty_0) \| + \| G(Tx_0) - G(Ty_0) \|^3}{\| G(Tx_0) - G(Ty_0) \|^2}$$
$$+\beta \frac{\| G(Ty_0) - T(Ty_0) \| \| G(Ty_0) - T(Tx_0) \| \| G(Tx_0) - T(Ty_0) \| + \| G(Tx_0) - G(Ty_0) \|^3}{\| G(Tx_0) - G(Ty_0) \|^2}$$
$$+\gamma \left[\| G(Tx_0) - T(Tx_0) \| + \| G(T(y_0)) - T(Ty_0) \| \right] + \delta \left[\| G(Tx_0) - T(Ty_0) \| + \| G(Ty_0) - T(Tx_0) \| \right]$$
$$+\eta \| G(Tx_0) - G(Ty_0) \|$$
$$= \alpha \| x_0 - y_0 \| + \beta \| x_0 - y_0 \| + 2\delta \| x_0 - y_0 \| + \eta \| x_0 - y_0 \|$$

i.e. $\| x_0 - y_0 \| \leq (\alpha + \beta + 2\delta + \eta) \| x_0 - y_0 \|$

But $\quad \alpha + \beta + 2\delta + \eta < 1$

So $\quad x_0 = y_0$

So common fixed point is unique.

Now we generalize the theorem (1.3) for three mappings.

Theorem 1.5. Let K be Closed and Convex subset of a Banach space X. Let $T : K \to G : K \to K$, $F : K \to K$ satisfies the following conditions.

$TG = GT$, $HT = TH$ and $GH = HG$ ... (1.5 *a*)

$T^2 = I$, $G^2 = I$, $H^2 = I$, ... (1.5 *b*)

where I denote the identity conditions

$$\| Tx - Ty \| \leq \alpha \frac{\| GH(x) - T(x) \| \| GH(y) - Ty \| \| GH(x) - Ty \| \| \| GH(x) - GH(y) \|^3}{\| GH(x) - GH(y) \|^2}$$
$$+\beta \frac{\| GH(y) - T(y) \| \| GH(y) - T(x) \| \| GH(x) - Ty \| + \| GH(x) - GH(y) \|^3}{\| GH(x) - GH(y) \|^2}$$
$$+\gamma \left[\| GH(x) - T(x) \| + \| GH(y) - T(y) \| \right] + \delta \left[\| GH(x) - Ty \| + \| GH(y) - Tx \| \right]$$
$$+\eta \| GH(x) - GH(y) \| \qquad \text{... (1.5 } c\text{)}$$

For every $x, y \in X$, $\alpha, \beta, \gamma, \delta, \eta \in [0,1]$, $x \neq y \in X$ with $\| GH(x) - GH(y) \| \neq 0$ and then there exists at least one common fixed point of T, G and H. If … with $10\alpha + 9\beta + 8\gamma + 5\delta + \eta < 4$ then T, H, G have unique common fixed point.

Proof : Suppose x is a point in Banach space X.

It is clear that $(TGH)^2 = I$

Now

$$\|TGH.G(x)-TGH.G(y)\|$$

$$\leq \alpha \frac{\|GH(GHG)(x)-T(GHG)(x)\|\,\|GH(GHG)(y)-T(GHG)(y)\|\,\|GH(GHG)(x)-T(GHG)(y)\|+\|GH(GHG)(x)-GH(GHG)(y)\|^3}{\|GH(GHG)(x)-GH(GHG)(y)\|^2}$$

$$+\beta \frac{\|GH(GHG)(y)-T(GHG)(y)\|\,\|GH(GHG)(y)-T(GHG)(x)\|\,\|GH(GHG)(x)-T(GHG)(y)\|+\|GH(GHG)(x)-GH(GHG)(y)\|^3}{\|GH(GHG)(x)-GH(GHG)(y)\|^2}$$

$$+\gamma\left[\|GH(GHG)(x)-T(GHG)(x)\|+\|GH(GHG)(y)-T(GHG)(y)\|\right]$$

$$+\delta\left[\|GH(GHG)(x)-T(GHG)(y)\|+\|GH(GHG)(y)-T(GHG)(x)\|\right]$$

$$+\eta\|GH(GHG)(x)-GH(GHG)(y)\|$$

$$=\alpha \frac{\|Gx-(TGH)G(x)\|\|\|G(y)-(TGH)G(y)\|\|\|Gx-(TGH)G(y)\|+\|G(x)-G(y)\|^3}{\|G(x)-G(y)\|^2}$$

$$+\beta \frac{\|G(y)-(TGH)G(y)\|\,\|G(y)-(TGH)G(x)\|\,\|Gx-(TGH)G(y)\|+\|G(x)-G(y)\|^3}{\|G(x)-G(y)\|^2}$$

$$+\gamma\left[\|Gx-(TGH)G(x)\|+\|G(y)-(TGH)G(y)\|\right]$$

$$+\delta\left[\|G(x)-(TGH)G(y)\|+\|G(y)-(TGH)G(x)\|\right]$$

$$+\eta\|G(x)-G(y)\|$$

We take $G(x) = p$ and $G(y) = q$

$$\|TGH(p)-TGH(q)\|$$

$$\leq \alpha \frac{\|p-TGH\|\,\|q-TGH(q)\|\,\|p-TGH(q)\|\|\|p-q\|^3}{\|p-q\|^2}$$

$$+\beta \frac{\|q-TGH(q)\|\,\|q-TGH(p)\|\,\|p-TGH(q)\|+\|p-q\|^3}{\|p-q\|^2} \qquad \text{... (1.5 } d\text{)}$$

$$+\gamma\left[\|p-T\,GH(p)\|+\|q-TGH(q)\|\right]+\delta\left[\|p-TGH(q)\|+\|q-TGH(p)\|\right]+\eta\|p-q\|$$

By previous theorem (1.3.*a*), (1.3.*b*) it is clear that TGH has at least one fixed point.

Say x_0 in K.

$$TGH(x_0)=x_0 \qquad \text{... (1.5 } e\text{)}$$

$$GH(TGH)(x_0)=GH(x_0) \qquad \text{... (1.5 } f\text{)}$$

$$(GH)^2\,T(x_0)=GH(x_0) \qquad \text{... (1.5 } g\text{)}$$

$$\Rightarrow \quad T(x_0)=GH(x_0) \qquad \text{... (1.5 } h\text{)}$$

Also $\quad H(TGH)(x_0)=H(x_0)$

$$TG(H)^2\,x_0=H(x_0)$$

$$\Rightarrow \quad TG(x_0)=H(x_0) \qquad \text{... (1.5 } i\text{)}$$

Now,

$$\| Hx_0 - x_0 \| \| TG(x_0) - T^2 \| = \| TG(x_0) - T(T(x_0)) \|$$

$$\le \alpha \frac{\| GH(G(x_0)) - T(G(x_0)) \| \, \| GH(T(x_0)) - T(T(x_0)) \| \, \| GH(G(x_0)) - T(T(x_0)) \| + \| GH(G(x_0)) - GH(T(x_0)) \|^3}{\| GH(G(x_0)) - GH(T(x_0)) \|^2}$$

$$+\beta \frac{\| GH(T(x_0)) - T(T(x_0)) \| \, \| GH(T(x_0)) - T(G(x_0)) \| \, \| GH(G(x_0)) - T(T(x_0)) \| + \| GH(G(x_0)) - GH(T(x_0)) \|^3}{\| GH(G(x_0)) - GH(T(x_0)) \|^2}$$

$$+\gamma\left[\| GH(G(x_0)) - T(G(x_0)) \| + \| GH(T(x_0)) - T(G(x_0)) \|\right]$$

$$+\delta\left[\| GH(G(x_0)) - T(T(x_0)) \| + \| GH(T(x_0)) - T(G(x_0)) \|\right]$$

$$+\eta \| GH(G(x_0)) - GH(T(x_0)) \|$$

$$= \alpha \| Hx_0 - x_0 \| + \beta \| Hx_0 - x_0 \| + \gamma\left[\| Hx_0 - Hx_0 \| + \| x_0 - x_0 \|\right]$$

$$+\delta\left[\| Hx_0 - x_0 \| + \| x_0 - Hx_0 \|\right] + \eta \| Hx_0 - x_0 \|$$

$$(\alpha + \beta + \gamma + 2\delta + \eta) \| Hx_0 - x_0 \|$$

i.e. $\| Hx_0 - x_0 \| < (\alpha + \beta + \gamma + 2\delta + \eta) \| Hx_0 - x_0 \|$

So, $Hx_0 = x_0$ because $\alpha + \beta + \gamma + 2\delta + \eta < 1$

So x_0 is fixed point of H.

Again

$$\| Tx_0 - x_0 \| = \| T(x_0) - T^2(x_0) \| = \| T(x_0) - T(Tx_0) \|$$

$$\le \alpha \frac{\| GH(x_0) - T(x_0) \| GH(Tx_0) - T(Tx_0) \| \, GH(x_0) - T(Tx_0) \| + \| GH(x_0) - GHT(x_0) \|^3}{\| GH(x_0) - GHT(x_0) \|^2}$$

$$+\beta \frac{\| GH(Tx_0) - T(Tx_0) \| \, \| GH(Tx_0) - T(x_0) \| \, GH(x_0) - TG(Tx_0) \| + \| GH(x_0) - GH(Tx_0) \|^3}{\| GH(x_0) - GH(Tx_0) \|^2}$$

$$+\gamma\left[\| GH(x_0) - T(Tx_0) \| + \| GH(Tx_0) - T(Tx_0) \|\right]$$

$$+\delta\left[\| GH(x_0) - T(Tx_0) \| + \| GH(Tx_0) - T(x_0) \|\right]$$

$$+\eta | \backslash GH(x_0) - GH(Tx_0) \|$$

$$= \alpha \| Tx_0 - x_0 \| + \beta \| Tx_0 - x_0 \| + \gamma\left[\| Tx_0 - x_0 \|\right] + \delta\left[\| T(x_0) - x_0 \| + \| x_0 - Tx_0 \|\right] + \eta \| Tx_0 - x_0 \|$$

$$= (\alpha + \beta + \gamma + 2\delta + \eta) \| Tx_0 - x_0 \| \text{ but } \alpha + \beta + \gamma + 2\delta + \eta < 1$$

So $T(x_0) = x_0$

Hence x_0 is fixed point of T.

But $G(x_0) = T(x_0) = x_0$

i.e. $G(x_0) = x_0$

Clearly x_0 is common fixed point of T, G and H.

Uniqueness: To show the uniqueness suppose $y_0 \ne x_0$ is another common fixed point of T, G and H so

$$\| x_0 - y_0 \| = \| T^2(x_0) - T^2(y_0) \| = \| T\,T(x_0) - T\,T(y_0) \|$$

$$\le \alpha \frac{\| GH(Tx_0) - T(Tx_0) \| \, \| GH(Ty_0) - T(Ty_0) \| \, \| GH(Tx_0) - T(Ty_0) \| + \| GH(Tx_0) - GH(Ty_0) \|^3}{\| GH(Tx_0) - GH(Ty_0) \|^2}$$

$$+\beta \frac{\| GH(Ty_0) - T(Ty_0) \| \, \| GH(Ty_0) - T(Tx_0) \| \, \| GH(Tx_0) - T(Ty_0) \| + \| GH(Tx_0) - GH(Ty_0) \|^3}{\| GH(Tx_0) - GH(Ty_0) \|^2}$$

$$+\gamma\left[\| GH(Tx_0) - T(Tx_0) \| + \| GH(Ty_0) - T(Ty_0) \|\right] + \delta\left[\| GH(Tx_0) - T(Ty_0) \| + \| GH(Ty_0) - T(Tx_0) \|\right]$$

$$+\eta \| GH(Tx_0) - GH(Ty_0) \|$$

$$=\alpha \| x_0 - y_0 \| + \beta \| x_0 - y_0 \| + \delta\left[\| x_0 - y_0 \| + \| x_0 - y_0 \|\right] + \eta \| x_0 - y_0 \|$$

$$\| x_0 - y_0 \| \leq \left(\alpha + \beta + \delta + \eta\right) \| x_0 - y_0 \|$$

So $x_0 = y_0$

Hence fixed point is unique.

REFERENCES

[1] Ahmad, A. and Shakil, M. "Some fixed point theorems in Banach spaces" *Nonlinear Funct. Anal. and Appl.* **11**: 343-349, (2006).

[2] Banach, S. "Surles operation dans les ensembles abstraits et leur application aux equations integrals" *Fund. Math.* **3**: 133-181, (1992).

[3] Badshah,V.H. and Gupta, O.P. "Fixed point theorems in Banach and 2- Banach spaces" *Jnanabha* **35**: 73-78, (2005).

[4] Brouwder, F.E. "Non-expansive non-linear operators in Banach spaces" *Proc. Nat. Acad. Sci. U.S.A.* **54**: 1041-1044, (1965).

[5] Datson, W.G.Jr. "Fixed point of quasi non-expansive mappings" *J, Austral. Math. Soc.* **13**: 167-172, (1972).

[6] Gohde, D. "Zum prinzip dev kontraktiven abbilduing" *Math. Nachr* **30**: 251-258, (1965).

[7] Goebel, K. "An elementary proof of the fixed point theorem of Browder and Kirk" *Michigan Math. J.* **16**: 381-383, (1969).

[8] Goebel, K. and Zlotkiewics, E. "Some fixed point theorems in Banach spaces" *Colloq Math* **23**: 103-106, (1971).

[9] Goebel, K. Kirk, W.A. and Shimt, T.N. "A fixed point theorem in uniformly convex spaces" *Boll. Un. Math, Italy* **4**: 67-75, (1973).

[10] Gahlar, S. "2- Metrche raume and ihre topologiscche structure" *Math. Nadh.* **26**: 115-148, (1963-64).

[11] Isekey, K. "fixed point theorem in Banach space" *Math Sem. Notes, Kobe University* **2**: 111-115, (1974).

[12] Jong, S.J. "Viscosity approximation methods for a family of finite non expansive in Banach spaces" *Nonlinear Analysis* **64**: 2536-2552, (2006).

[13] Khan, M.S. "Fixed points and their approximation in Banach spaces for certain commuting mappings" *Glasgow Math. Jour.* **23**: 1-6, (1982).

[14] Khan, M.S. and Imdad, M. "Fixed points of certain involutions in Banach spaces" *J. Austral. Math. Soc.* **37**: 169-177, (1984).

[15] Kirk, W.A. "A fixed point theorem mappings do not increase distance" *Amer. Math. Monthly* **72**: 1004-1006, (1965).

[16] Kirk, W.A. "A fixed point theorem for non-expansive mappings" Lecture notes in Math. Springer-Verlag, Berlin and New York **886**: 111-120, (1981).

[17] Kirk, W.A "Fixed point theorem for non-expansive mappings" *Contem Math.* **18**: 121-140, (1983).

[18] Pathak, H.K. and Maity, A.R. "A fixed point theorem in Banach space" *Acta Ciencia Indica* **17**: 137-139, (1991).

[19] Qureshi, N.A. and Singh, B. "A fixed point theorem in Banach space" *Acta Ciencia Indica* **11**: 282-284, (1995).

[20] Rajput, S.S. and Naroliya, N. " Fixed point theorem in Banach space" *Acta Ciencia Indica* **17**: 469-474, (1991).

[21] Sharma, P.L. and Rajput, S.S. " Fixed point theorem in Banach space" *Vikram Mathematical Journal* **4**: 35-38, (1983).

[22] Singh, M.R. and Chatterjee, A.K. "Fixed point theorem in Banach space" *Pure Math. Manuscript* **6**: 53-61, (1987).

[23] Sharma, S. and Bhagwan, A. "Common fixed point theorems on Normed space" *Acta Ciencia Indica* **31**: 20-24, (2003).

[24] Shahzad, N and Udomene, A. "Fixed point solutions of variational inequalities for asymptotically non-expansive mappings in Banach spaces" *Nonlinear Analysis* **64**: 558-567, (2006)

[25] Verma, B.P. "Application of Banach fixed point theorem to solve non linear equations and its generalization" *Jnanabha* **36**: 21-23, (2006).

[26] Yadava, R.N., Rajput, S.S. and Bhardwaj, R.K. "Some fixed point and common fixed point theorems in Banach spaces" *Acta Ciencia Indica 33* No. **2**: 453-460, (2007).

[27] Yadava, R.N., Rajput, S.S., Choudhary, S. and Bhardwaj, R.K. "Some fixed point and common fixed point theorems for non-contraction mapping on 2-Banach spaces" *Acta Ciencia Indica 33* No. **3**: 737-744, (2007).