International Journal of Theoretical and Applied Science **3**(2): 88-90(2011)

ISSN No. (Print) : 0975-1718
ISSN No. (Online) : 2249-3247

Some Optimization Problems Involving Moments of Discrete Random Variables

P.L. Sharma and R. Sharma

Department of Mathematics, H.P University, Shimla, (HP)

(Received 15 September, 2011, Accepted 20 October, 2011)

ABSTRACT : We use Lagrange multiplier method to give an alternative proof of the inequality involving moments of a discrete random variable. We also discuss an alternative proof of the inequality between arithmetic mean and variance of discrete uniform distributions.

Keywords: Moments ,discrete distribution ,variance , Lagrange multipliers.

I. INTRODUCTION

Let $\{p_1, p_2, ..., p_n\}$ be the probability distribution with support $\{x_1, x_2, ..., x_n)$. The r^{th} order moment μ'_r is defined as

$$\mu'_r = \sum_{i=1}^{n} p_i x_i^r \qquad ...(1.1)$$

The inequalities between the moments of the discrete probability distributions have been studied extensively in literature. It is shown that the Lagrange and Kuhn Tucker methods are useful in investigating such inequalities, see [1-2]. The variance upper bounds are important in the field of theory of mathematical statistics. A number of important inequalities exist in literature, for more details see [3-10].

In the present paper, we first derive an inequality involving moments of discrete probability distributions (theorem 2.1, below). We show a connection between an inequality due to Muilwijk [10] and Mohr's circle diagram in the theory of elasticity, (Lemma 2.2, below). It follows from Mohr's circle diagram that the Muilwijk inequality is true for $n = 3$, we then show on using the similar analysis that the inequality must be true for n_i (Theorem 2.3, below) also see [11].

II. MAIN RESULTS

Theorem 2.1. Under the above notations:

$$\mu'_3 \geq \mu'_1 \mu'_2 \qquad ... (2.1)$$

If $x_i > 0$, $i = 1, 2, ..., n$, then

$$\mu'_3 \geq \frac{\mu_2'^2}{\mu_1} \qquad ... (2.2)$$

Proof : We minimize the function

$$f(x) = \sum_{t-1}^{n} p_t x_t^3 \qquad ... (2.3)$$

Subject to the constraints

$$g_1(x) = \sum_{i=1}^{n} p_i x_i^2 - k_1 \qquad ... (2.4)$$

and

$$g_2(x) = \sum_{i=2}^{n} p_i x_i^2 - k_2 \qquad ... (2.5)$$

The Lagrange function is

$$l(x,\lambda) = \sum_{i=1}^{n} p_i x_i^3 - \lambda_1\left(\sum_{i=1}^{n} p_i x_i^2 - k_1\right) - \lambda_2\left(\sum_{i=1}^{n} p_i x_i - k_2\right) \qquad ... (2.6)$$

The derivatives are

$$\frac{\partial L}{\partial x_i} = (3x_i^2 - 2\lambda_1 x_i - \lambda_2)\, p_i \qquad ... (2.7)$$

$$\frac{\partial L}{\partial \lambda_1} = k_1 - \sum_{i=1}^{n} p_i x_i^2 \qquad ... (2.8)$$

and

$$\frac{\partial L}{\partial \lambda_2} = k_2 - \sum_{i=1}^{n} p_i x_i \qquad ... (2.9)$$

The solutions of these equations

$$\frac{\partial L}{\partial x_i} = 0, \frac{\partial L}{\partial \lambda_2} = 0 \text{ and } \frac{\partial L}{\partial \lambda_2} = 0 \qquad ... (2.10)$$

give $\qquad x_i = k_2 \qquad ... (2.11)$

as $\frac{\partial L}{\partial x_i} = 0$ implies that all x_i are equal, $i = 1, 2, ..., n$. Also

$$k_1 = k_2^2 \qquad ... (2.12)$$

For $x_i \geq 0$, $i = 1, 2, ..., n$, the Hessian matrix

$$\begin{bmatrix} 6p_1x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 6p_nx_n \end{bmatrix} \qquad ... (2.13)$$

is positive definite, therefore the function is convex. So $x_i = k_2$ gives the minimum of $f(x)$. Hence

$$f(x) = \sum_{i=1}^{n} p_i x_i^3 \geq k_2^3 \qquad ... (2.14)$$

Since $k_1 = k_2^2$, therefore from (2.14), we have

$$f(x)-\sum_{i=1}^{n} p_i x_i^3 \geq k_2^2 k_2 - k_1 k_2 \quad \text{... (2.15)}$$

Also $\frac{\partial L}{\partial \lambda_1}=0$ and $\frac{\partial L}{\partial \lambda_2}=0$ respectively gives $k_1 = \mu'_1\mu'_2$. The inequality (2.1) now follows from (2.15).

From (2.11) and (2.12), we get

$$x_i = \frac{k_1}{k_2}, i = 1, 2, ..., n \quad \text{... (2.16)}$$

Therefore we have

$$f(x) = \sum_{i=1}^{n} p_i x_i^3 \geq \frac{k_1^3}{k_2^3} = \frac{k_1^2}{k_2} \quad \text{... (2.17)}$$

The inequality (2.2) follows from (2.17).

Lemma 2.2. Let $a \leq x_i \leq b$, $i = 1, 2, 3, \ldots, n$. For $n = 3$, we have

$$\mu'_2 \leq (a + b)\mu'_1 - ab \quad \text{... (2.18)}$$

and

$$\mu'_2 \geq (x_{j-1} + x_j)\mu'_1 - x_{j-1}x_j \quad \text{... (2.19)}$$

$j = 2, 3$. The inequalities (2.18) and (2.19) become equalities when $n = 2$.

Proof : We have

$$p_1 + p_2 + p_3 - 1 \quad \text{... (2.20)}$$

$$x_1 p_1 + x_2 p_2 + x_3 p_3 = \mu'_1 \quad \text{... (2.21)}$$

and

$$x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 = u'_2 \quad \text{... (2.22)}$$

The solution of the simultaneous system of linear equations is

$$p_1 = \frac{\mu'_2 - (x_2 + x_3)\mu'_1 + x_2 x_3}{(x_2 - x_1)(x_3 - x_1)} \quad \text{... (2.23)}$$

$$p_2 = \frac{\mu'_2 - (x_1 + x_3)\mu'_1 + x_1 x_3}{(x_2 - x_1)(x_2 - x_3)} \quad \text{... (2.24)}$$

and

$$p_3 = \frac{\mu'_2 - (x_1 + x_2)\mu'_1 + x_1 x_2}{(x_3 - x_2)(x_3 - x_1)} \quad \text{... (2.25)}$$

For $x_1 < x_2 < x_3$, we have $(x_2 - x_1)(x_3 - x_1) > 0$, also $p_1 \geq 0$ therefore it follows from (2.23) that

$$\mu'_2 \geq (x_2 + x_3)\mu'_1 - x_2 x_3 \quad \text{... (2.26)}$$

Similarly, on using similar arguments, it follows from (2.25) that

$$\mu'_2 \geq (x_1 + x_2)\mu'_1 - x_1 x_2 \quad \text{... (2.27)}$$

Likewise, the inequality (2.18) follows from (2.24). Further, it follows from direct calculations that for n = 2, we have $u'_2 = (x_1 + x_2)u'_1 - x_1 x_2$.

Theorem 2.2. For real numbers $x_1, x_2, \ldots, x_n$, we have

$$\mu'_2 \geq (a + b)\mu'_1 - ab \quad \text{... (2.28)}$$

and

$$\mu'_2 \geq (x_{j-1} + x_j)\mu'_1 - x_{j-1}x_j \quad \text{... (2.29)}$$

$j = 2, 3, \ldots, n$.

Proof : By Lemma 2.2, the theorem is true for $n = 3$. For $n \geq 4$, we write

$$p_\alpha + p_\beta + p_\gamma = 1 - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i \quad \text{... (2.30)}$$

$$p_\alpha x_\alpha + p_\beta x_\beta + p_\gamma x_\gamma = \mu'_1 - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i x_i \quad \text{... (2.31)}$$

and

$$x_\alpha^2 p_\alpha + x_\beta^2 p_\beta + x_\gamma^2 p_\gamma = \mu'_2 - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i x_i^2 \quad \text{... (2.32)}$$

The solution of the system of the linear equations (2.30), (2.31) and (2.32) can be written as

$$p_\alpha = \frac{\mu'_2 - (x_\beta + x_\gamma)\mu'_1 + x_\beta x_\gamma - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i (x_i - x_\beta)(x_i - x_\gamma)}{(x_\beta - x_\alpha)(x_\gamma - x_\alpha)} \quad \text{... (2.33)}$$

$$p_\beta = \frac{u'_2 - (x_\alpha + x_\gamma)u'_1 + x_\alpha x_\gamma - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i (x_i - x_\alpha)(x_i - x_\gamma)}{(x_\beta - x_\alpha)(x_\beta - x_\gamma)} \quad \text{... (2.34)}$$

and

$$p_\gamma = \frac{u'_2 - (x_\alpha + x_\beta)u'_1 + x_\alpha x_\gamma - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i (x_i - x_\alpha)(x_i - x_\beta)}{(x_\gamma - x_\alpha)(x_\gamma - x_\beta)} \quad \text{... (2.35)}$$

where α, β and γ take values $1, 2, \ldots, n$ with $\alpha \neq \beta \neq \gamma$. Let $\alpha = 1$ and $\gamma = n$. From (2.34) we have

$$p_\beta = \frac{\mu'_2 - (x_1 + x_n)\mu'_1 + x_1 x_n - \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i (x_i - x_1)(x_i - x_n)}{(x_\beta - x_1)(x_\beta - x_n)} \quad \text{... (2.36)}$$

For $x_1 \leq x_\beta \leq x_n$ we have $(x_\beta - x_1)(x_\beta - x_n) \leq 0$. Also $p_\beta \geq 0$, therefore it follows from (2.36) that

$$\mu'_2 - (x_1 + x_n)\mu'_1 + x_\alpha x_\beta \geq \sum_{i \neq \alpha,\beta,\gamma}^{n} p_i (x_i - x_1)(x_i - x_n) \quad \text{... (2.37)}$$

Since $(x_i - x_1)(x_i - x_n) \leq 0$, for $i = 1, 2, \ldots, n$ therefore the inequality (2.28) follows from (2.37).

We now consider the case when α, β and γ take consecutive values. So $x_\alpha \leq x_\beta \leq x_\gamma$ and $(x_\gamma - x_\beta)(x_\gamma - x_\alpha) \geq 0$. Since $p_\gamma \geq 0$, therefore from (2.35)

$$\mu'_2 - (x_\alpha + x_\beta)\mu'_1 + x_\alpha x_\beta \geq \sum_{i=1, i \neq \beta}^{n-1} p_i (x_i - x_\alpha)(x_i - x_\beta) \quad \text{... (2.38)}$$

The inequality (2.29) therefore follows from (2.38), as $(x_i - x_\alpha)(x_i - x_\beta) \geq 0$ for $1, 2, \ldots, n - 1$.

Remark : If S^2 be the variance of real numbers $x_1, x_2, \ldots, x_n$, then $\mu'_2 = S^2 + \mu_1'^2$. The Muilwijk inequality, namely, $S^2 \leq (b - \mu'_1)(u'_1 - a)$, follows from the inequality (2.28).

Acknowledgements: Authors acknowledge the support of UGC-SAP .

REFERENCES

[1] Sharma, P.L. and Sharma, R., A note on inequalities between the moments, *International Journal of Theoretical and Applied Sciences*, **2**(2): 30-33, (2010).

[2] Sharma, P.L. and Sharma, R. Some optimization problems related with inequalities between the moments, *Bulletin of Pure and Applied Sciences*, **30**(1): 143-152, (2011).

[3] Shohat, J.A. and Tamarkin, J.D., Mathematical Surveys Number I. *The problem Moments, American mathematical society*, (1963).

[4] Mitrinovic D.S. Analytic Inequalities, Springer Verlag-New York, (1970).

[5] Kendall, M. and Stuart, A., *The Advanced Theory of Statistics*, Vol. **1**: 4th edition, Charles Grafin and Co. London, (1977).

[6] Samuelson, P.A., How deviant can you be? *Journal of the American Statistical Association*, **63**: 1522-1525, (1968).

[7] Jensen, S.T. M.Sc. Thesis, The Laguerre Samuelson Inequality with extensions and applications in Statistics and Matrix Theory, Department of Mathematics and Statistics, McGill University, Montreal, Canada, (1999).

[8] Brunk, H.D., Note on two papers of K.R. Nair, *Journal of the Indian Society of Agricultural Statistics*, **11**: 186-189, (1959).

[9] Murthy, M.N. and Sethi, V.K., Self-Weighting design at Jabulation stage, Sankhya Ser, B27, 201-210, (1965).

[10] Muilwijk, J., Note on a theorem of M.N. Murthy and V.K. Sethi, Sankhya Ser, B28, 183, (1966).

[11] Sharma, R. and Ranjan, A., Elastic properties of reinforced solid: Analogus of Mohr Circle diagram-II, *Journal of Reinforced Plastics and Composites*, **21**(14): 1319-1336, (2002).