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Fuzzy approach with linear and some non-linear membership functions for solving multi-objective assignment problems

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Abstract- Assignment Problem (A.P.) originates from the classical problems where the objective is to find the optimum assignment of a number of jobs (tasks) to an equal number of machines (or workers) at a minimum cost or minimum time. The multi-objective assignment problem refers to a special class of vector minimum linear programming problems. In this paper, we use a special type of linear and non-linear membership functions to solve the multi-objective assignment problem. It gives an optimal compromise solution. The result obtained by using a linear membership function has been compared with the solution obtained by using non-linear membership functions. Numerical example has been provided to illustrate the solution procedure.

Keywords- Assignment Problem, Multi-criteria decision making, Linear membership function, Non-linear membership function

Introduction

The Assignment Problem (A.P.) is one of the most-studied, well-known and important problem in mathematical programming in which our objective is to assign a number of jobs (tasks) to an equal number of machines (workers) so as to minimize the total assignment cost or to minimize the total consumed time for execution of all the jobs (tasks). Hence assignment problem can be viewed as a balanced transportation problem, in which all supplies and demands equal to 1, and the number of rows and columns in the matrix are identical. Hence, Ravindran et al [6] can be used the transportation simplex method to solve the assignment problems. However, as an assignment problem is highly degenerate it will be frustrating or inefficient and not recommended to attempt to solve it by simplex method. Another technique called Hungarian method is commonly employed to solve the minimizing assignment problem by Ravindran et al [7]. Geetha et al [3] first expressed the cost-time minimizing assignment as the multicriteria problem. Bit et al.[1] applied the fuzzy programming technique with linear membership function to solve the multi-objective transportation problem. Tsai et al [10] provided a solution for balanced multi-objective decision making problem associated with cost, time and quality by fuzzy concept. The Linear Interactive and Discrete Optimization (LINDO) Schrage [8], General Interactive Optimizer (GINO) Liebman[5] and TORA packages Taha[9] as well as many other commercial and academic packages are useful to find the solution of the assignment problem. Zadeh [12] first introduced the concept of fuzzy set theory. Then, Zimmermann [13] first applied suitable membership functions to solve linear programming problem with several objective functions. He showed that solutions obtained by fuzzy linear programming are always efficient. Leberling [4] used a special-type non-linear membership functions for the vector maximum linear programming problem. He showed that solutions obtained by fuzzy linear programming with this type of non-linear membership functions are always efficient. Verma et al [11] used the fuzzy programming technique with some non-linear (hyperbolic and exponential) membership functions to solve a multi-objective transportation problem are always efficient. Dhingra et al [2] defined other types of the non-linear membership functions and applied them to an optimal design problem. In the multi-objective assignment problem, only the objectives are considered as fuzzy. We apply the fuzzy approach with linear and some non-linear membership functions to solve a multi-objective assignment problem as a vector minimum problem.

Assumptions and notations

The following assumptions and notations are used in developing the proposed model:

There are n jobs (tasks) in a factory and the factory has n machines (workers) to process the jobs (tasks).

Each job can be associated with one and only one machine.

Penalty $C_{ij} \geq 0$ be the execution cost, time etc which is incurred when a job i(i=1,2,…,n) is processed by the machine j(j=1,2,…,n).

The crisp number X_{ij} denotes that the i^{th} job is assigned to the j^{th} machine.

Each machine can perform each job but with varying degree of efficiency.

Mathematical Formulation

A Multi-objective assignment problem may be stated mathematically as:

$$\text{Minimize } Z_k = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{k} X_{ij} \right. \quad , \ k = 1,2,\ldots,K$$

or

$$\text{Minimize } Z_k = \begin{cases} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{1} X_{ij} \\ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{2} X_{ij} \\ . \\ . \\ . \\ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{k} X_{ij} \end{cases} \tag{1}$$

subject to

$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j = 1,2,\ldots,n \tag{2}$$

$$\sum_{j=1}^{n} X_{ij} = 1, \qquad i = 1,2,\ldots,n \tag{3}$$

$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases} \tag{4}$$

The constraint (2) ensures that only one job is assigned to one machine while the constraint (3) ensures that only one machine should be assigned to one job. And the subscript on Z_k and superscript on c_{ij}^{k} denote the k^{th} penalty criterion.

Fuzzy approach for the multi-objective assignment problem

The Multi-objective assignment problem can be considered as a vector minimum problem. The first step is to assign, for each objective, two values U_k and L_k as upper and lower bounds for the objective function Z_k: L_k=Aspired level of achievement for objective k, U_k= Highest acceptable level of achievement for objective k and d_k= U_k-L_k the degradation allowance for objective k. Once the aspiration levels and degradation for each objective have been specified, we have formed the fuzzy model. Our next step is to transform the fuzzy model into a 'crisp' model.

Algorithm

Step 1: Solve the Multi-objective assignment problem as a single objective assignment problem k times by taking one of the objectives at a time.

Step 2: From the results of step 1, determine the corresponding values for every objective at each solution derived. According to each solution and value for every objective, we can find pay-off matrix as follows:

$$
\begin{array}{c c}
 & \begin{array}{cccc} Z_1(X) & Z_2(X) & \ldots & Z_k(X) \end{array} \\
\begin{array}{c} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(k)} \end{array} &
\begin{bmatrix} Z_{11} & Z_{12} & \ldots & Z_{1k} \\ Z_{21} & Z_{22} & \ldots & Z_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{k1} & Z_{k2} & \ldots & Z_{kk} \end{bmatrix}
\end{array}
$$

Where $X^{(1)}, X^{(2)}, \ldots, X^{(k)}$ are the isolated optimal solutions of the k different assignment problems for k different objective functions. $Z_{ij}=Z_j(X^i)$ $(i=1,2,\ldots,k \;\; \& \; j=1,2,\ldots,k)$ be the i-th row and j-th column element of the pay-off matrix.

Step 3: From step 2, we find for each objective the worst (U_k) and the best (L_k) values corresponding to the set of solutions, where,

$U_k = \max(Z_{1k}, Z_{2k}, \ldots, Z_{kk})$ and

$L_k = Z_{kk} \qquad k=1,2,\ldots,K$

Step 4: Define membership functions (linear μ or hyperbolic μ^H or exponential μ^E) for the k-th objective function as follows:

Case (i) A linear membership function for the k-th objective function is defined by $\mu_k(X)$ and shown in Fig. (1).

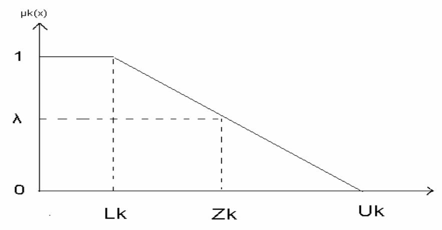

Fig. 1- The linear membership function

$$
\mu_k(x) = \begin{cases} 1, & \text{if } Z_k \le L_k \\ 1 - \dfrac{Z_k - L_k}{U_k - L_k}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \ge U_k \end{cases} \tag{5}
$$

Case (ii) An hyperbolic membership function for the k-th objective function is defined by $\mu_{Z_k}^{H}(x)$

$$
\mu_{Z_k}^{H}(x) = \begin{cases} 1, & \text{if } Z_k \le L_k \\ \dfrac{1}{2}\dfrac{e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k} - e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k}}{e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k} + e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k}} + \dfrac{1}{2}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \ge U_k \end{cases} \tag{6}
$$

Where, $\alpha_k = 6/\left(U_k - L_k\right)$

Case (iii) An exponential membership function for the k^{th} objective function is defined by $\mu_{Z_k}^{E}(x)$

$$
\mu_{Z_k}^{E}(x) = \begin{cases} 1, & \text{if } Z_k \le L_k \\ \dfrac{e^{-S\Psi_k(X)} - e^{-S}}{1 - e^{-S}}, & \text{if } L_k < Z_k < U_k \\ 0, & \text{if } Z_k \ge U_k \end{cases} \tag{7}
$$

Where $\Psi_k(X) = \dfrac{Z_k - L_k}{U_k - L_k} \qquad k=1,2,\ldots,K$

S is a non-zero parameter, prescribed by the decision maker.

Step 5: From step 4, we can find an equivalent crisp model for the initial fuzzy model as follows:

If we will use the linear membership function as defined in (5) then an equivalent crisp model for the fuzzy model can be formulated as:

Maximize λ

subject to

$$\lambda \le \frac{U_k - Z_k(X)}{U_k - L_k}, \quad k=1,2,\ldots,K$$

$$\sum_{i=1}^{n} X_{ij} = 1, \quad j=1,2,\ldots,n\,; \quad \sum_{j=1}^{n} X_{ij} = 1, \quad i=1,2,\ldots,n\,; \quad \lambda \ge 0$$

$$X_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The above problem can be further simplified as:

$$\text{Maximize } \lambda$$
$$\text{subject to}$$
$$Z_k(X) + \lambda(U_k - L_k) \le U_k \qquad ,k=1,2,...,K$$
$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j=1,2,...,n; \quad \sum_{j=1}^{n} X_{ij} = 1, \qquad i=1,2,...,n; \quad \lambda \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

If we use a hyperbolic membership function as defined in (6) then an equivalent crisp model for the fuzzy model can be formulated as:

$$\text{Maximize } \lambda$$
$$\text{subject to}$$
$$\lambda \le \frac{1}{2}\,\frac{e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k} - e^{-\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k}}{e^{\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k} + e^{-\left\{\frac{(U_k+L_k)}{2} - Z_k(x)\right\}\alpha_k}} + \frac{1}{2} \quad ,k=1,2,...,K$$
$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j=1,2,...,n; \quad \sum_{j=1}^{n} X_{ij} = 1, \qquad i=1,2,...,n; \quad \lambda \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The above problem can be further simplified as

$$\text{Maximize } X_{nn+1}$$
$$\text{subject to}$$
$$\alpha_k\, Z_k(x) + X_{nn+1} \le \alpha_k (U_k + L_k)/2, \qquad k = 1,2,-----K$$
$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j=1,2,...,n; \quad \sum_{j=1}^{n} X_{ij} = 1, \qquad i=1,2,...,n; \quad X_{nn+1} \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

Where, $X_{nn+1} = \tanh^{-1}(2\lambda - 1)$

If we use exponential membership function as defined in (7) then an equivalent crisp model for the fuzzy model can be formulated as:

$$\text{Maximize } \lambda$$
$$\text{subject to}$$
$$\lambda \le \frac{e^{-S\Psi_k(X)} - e^{-S}}{1 - e^{-S}} \quad ,k=1,2,...,K$$
$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j=1,2,...,n; \quad \sum_{j=1}^{n} X_{ij} = 1, \qquad i=1,2,...,n; \quad \lambda \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The above problem can be further simplified as:

$$\text{Maximize } \lambda$$
$$\text{subject to}$$
$$e^{-S\Psi_k(X)} - (1 - e^{-S})\lambda \ge e^{-S} \quad ,k=1,2,...,K$$
$$\sum_{i=1}^{n} X_{ij} = 1, \qquad j=1,2,...,n; \quad \sum_{j=1}^{n} X_{ij} = 1, \qquad i=1,2,...,n; \quad \lambda \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

Step 6: Solve the crisp model by an appropriate mathematical programming algorithm.

The solution obtained in step 6 will be the optimal compromise solution of the Multi-objective assignment problem.

Numerical Example

$$\text{Minimize: } Z_1 = 10X_{11}+8X_{12}+15X_{13}+13X_{21}+12X_{22}+13X_{23}+8X_{31}+10X_{32}+9X_{33}$$
$$\text{Minimize: } Z_2 = 13X_{11}+15X_{12}+8X_{13}+10X_{21}+20X_{22}+12X_{23}+15X_{31}+10X_{32}+12X_{33}$$
$$\text{Subject to}$$
$$\sum_{i=1}^{3} X_{ij} = 1, \qquad j=1,2,3; \quad \sum_{j=1}^{3} X_{ij} = 1, \qquad i=1,2,3$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

For the objective Z_1, we find the optimal solution as

$$X^{(1)} = \begin{cases} X_{12} = 1,\ X_{23} = 1,\ X_{31} = 1, \\ \text{and rest all } X_{ij}\text{'s are zeros} \end{cases}$$

and $Z_1 = 29$

For the objective Z_2, we find the optimal solution as

$$X^{(2)} = \begin{cases} X_{13} = 1,\ X_{21} = 1,\ X_{32} = 1, \\ \text{and rest all } X_{ij}\text{'s are zeros} \end{cases}$$

and $Z_2 = 28$

We can write the payoff matrix as

$$\begin{array}{c c} & \begin{array}{cc} Z_1(X) & Z_2(X) \end{array} \\ \begin{array}{c} X^{(1)} \\ X^{(2)} \end{array} & \begin{bmatrix} 29 & 38 \\ 42 & 28 \end{bmatrix} \end{array}$$

From the pay-off matrix we find the upper bound and lower bound

U_1= max (29, 38) = 38, U_2= max (42, 28) = 42, L_1= 29, L_2= 28, d_1 = 9, d_2 = 14

If we use the linear membership function as defined in(5), an equivalent crisp model can be formulated as:

$$\text{Maximize } \lambda$$
$$\text{Subject to}$$
$$10X_{11}+8X_{12}+15X_{13}+13X_{21}+12X_{22}+13X_{23}+8X_{31}+10X_{32}+9X_{33}+9\lambda \le 38$$
$$13X_{11}+15X_{12}+8X_{13}+10X_{21}+20X_{22}+12X_{23}+15X_{31}+10X_{32}+12X_{33}+14\lambda \le 42$$
$$\sum_{i=1}^{3} X_{ij} = 1, \qquad j=1,2,3; \quad \sum_{j=1}^{3} X_{ij} = 1, \qquad i=1,2,3; \quad \lambda \ge 0$$
$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The problem is solved by the linear interactive and discrete optimization (LINDO) software. The optimal solution is presented as follows:

$$X^* = \begin{cases} X_{12} = 1,\ X_{21} = 1,\ X_{33} = 1, \\ \text{and rest all } X_{ij}\text{'s are zeros} \end{cases}$$

$$Z_1^* = 30,\ Z_2^* = 37 \text{ and } \lambda = 0.58$$

If we use the hyperbolic membership function as defined in (6), an equivalent crisp model can be formulated as:

$$\text{Maximize } X_{10}$$

Subject to

$$60X_{11}+48X_{12}+90X_{13}+78X_{21}+72X_{22}+78X_{23}+48X_{31}+60X_{32}+54X_{33}+9X_{nn+1} \le 201$$

$$78X_{11}+90X_{12}+48X_{13}+60X_{21}+120X_{22}+72X_{23}+90X_{31}+60X_{32}+72X_{33}+14X_{nn+1} \le 210$$

$$\sum_{i=1}^{3} X_{ij}=1, \quad j=1,2,3; \sum_{j=1}^{3} X_{ij}=1, \quad i=1,2,3; X_{10} \ge 0$$

$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The problem is solved by the linear interactive and discrete optimization (LINDO) software. The optimal solution is presented as follows:

$$X^* = \begin{cases} X_{12}=1,\ X_{21}=1,\ X_{33}=1, \\ \text{and rest all } X_{ij}\text{'s are zeros} \end{cases}$$

$X_{nn+1}=0.4818653$

But here,

$X_{nn+1}=\tanh^{-1}(2\lambda-1)$

$\tanh(0.4818653)= 2\lambda-1$

$\lambda=0.50$

Therefore

$$Z_1^* = 30,\ Z_2^* = 37 \text{ and } \lambda = 0.50$$

However, If we use exponential membership function as defined in (7) with the parameter S=1, an equivalent crisp model for the fuzzy model can be formulated as:

$$\text{Maximize } \lambda$$

Subject to

$$\exp\{(-10X_{11}-8X_{12}-15X_{13}-13X_{21}-12X_{22}-13X_{23}-8X_{31}-10X_{32}-9X_{33}+29)/9\}$$
$$-0.6321205\lambda \ge 0.3678794$$

$$\exp\{(-13X_{11}-15X_{12}-8X_{13}-10X_{21}-20X_{22}-12X_{23}-15X_{31}-10X_{32}-12X_{33}+28)/14\}$$
$$-0.6321205\lambda \ge 0.3678794$$

$$\sum_{i=1}^{3} X_{ij}=1, \quad j=1,2,3; \sum_{j=1}^{3} X_{ij}=1, \quad i=1,2,3; \lambda \ge 0$$

$$X_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \\ 0, \text{ if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \end{cases}$$

The problem is solved by the General Interactive Optimizer (GINO) software. The optimal solution is presented as follows:

$$X^* = \begin{cases} X_{12}=1,\ X_{21}=1,\ X_{33}=1, \\ \text{and rest all } X_{ij}\text{'s are zeros} \end{cases}$$

$$Z_1^* = 30,\ Z_2^* = 37 \text{ and } \lambda = 0.45$$

Conclusion

In this paper, linear and non-linear membership functions have been used to solve the multi-objective assignment problem. If we use the hyperbolic membership function, then the crisp model becomes linear. The optimal compromise solution does not change significantly if we compare with the solution obtained by the linear membership function. However, if we use the exponential type membership function, with different values of S (parameter) then the crisp model becomes non-linear and the optimal compromise solution does not change significantly, if we compare with the solution obtained by the linear membership function.

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