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Recognition of Convex Bodies by Covariograms

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Abstract. The present paper contains a review of the main results of Yerevan research group in tomography of planar bounded convex domains. The applications of these problems are known in both geometric and computer tomography. Complicated geometrical patterns occur in many areas of science. Their analysis requires creation of mathematical models and development of special mathematical tools. The corresponding area of mathematical research is called Stochastic Geometry. Among more popular applications are Stereology and Tomography. The methods of form analysis are based on analysis of the objects as figures, i.e. as subsets of the plane. For these sets, geometrical characteristics are considered that are independent of the position and orientation of the figures (hence they coincide for congruent figures). Classical examples are area and perimeter of a figure. In the last century German mathematician W. Blaschke formulated the problem of investigation of bounded convex domains in the plane using probabilistic methods. In particular, the problem of recognition of bounded convex domains by chord length distribution.

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Is there a one-to-one correspondence between bounded convex bodies D and their chord length distribution functions $F_D(x)$? It was an old question of W. Blaschke whether the random chord length determine the convex body D uniquely, up to a rigid motion (see **[1]**). This was disproved by Mallow and Clark (see **[2]**), who constructed two non-congruent bounded convex 12-gons with the same chord length distribution. Therefore, chord length distribution function $F_D(x)$ does not reconstruct convex bodies. One possible way of treating this problem is to consider subclasses of the class of convex bodies for which the chord length distribution provides sufficient information to distinguish between non-congruent members (see **[3]** and **[4]**). Gates (see **[3]**) showed that triangles and quadrangles can be reconstructed from their chord length distributions.

Another method consists of the consideration of chord length measurements not in the completely mixed form of the distribution but in preserving information about the line which generates the chord (see [**5**] and [**10**]). If for any line its precise location as well as the chord length is known, that is the Radon transform occurs, the conditions under which sets can be reconstructed uniquely have been studied in depth. The applications in both geometric and computer tomography are well known (see **[6]** and [**24**]). For example, we can consider the case that the orientation and the length of the chords are observed. We refer to it as the orientation-dependent chord length distribution, that is for any fixed direction the distribution of the chord lengths is considered.

A different question arises if one considers, for each direction $u \in S^{n-1}$ (S^{n-1} is the sphere of unit radius centered at the origin) the distribution of the length of the uniform random chord of the convex body D with direction u. Let D be a convex body in n-dimensional Euclidean space R^n, that is a compact, convex subset of R^n, with non-empty interior. The n-dimensional Lebesgue measure in R^n is denoted by $L_n(\cdot)$. If $h \in R^n$, then D+h denotes the translate of D by h, that is $D + h = \{x + h,\ h \in R^n,\ x \in D\}$.

The covariogram of a convex body $D \subset R^n$ is the function $C(D,h): R^n \to [0,\infty]$, defined for $h \in R^n$ by $C(D,h) = L_n\big(D \cap (D+h)\big)$.

The covariogram C(D, h) is clearly unchanged with respect to translations and reflections of D. This function was introduced by G. Matheron in his book **[8]** on random sets. In **[9]** G. Matheron asked the following question and conjectured that a planar convex body is uniquely determined (within the class of convex bodies) by its covariogram, up to translation and reflection.

In the plane, an affirmative answer to the covariogram problem for convex polygons was given by Nagel (see **[10]**). Various partial results were obtained by several authors (see the references in [**11**] and [**12**]), until Averkov and Bianchi (see **[12]**) finally settled the problem completely for arbitrary convex bodies in the plane: Every planar convex body is determined within all planar convex bodies by its covariogram, up to translations and reflections.

Very little is known regarding the covariogram problem when the space dimension is larger than 2. It is known that centrally symmetric convex bodies in any dimension, are determined by their covariogram, up to translations. This is a consequence of the fact that $C(D,h)$ determines the volume of D (=C(D, 0)) and its difference body D-D and of the Brunn-Minkowski inequality. This inequality implies that among all convex bodies with the same difference body the centrally symmetric one is the only set of the maximal volume (see **[6]**).

Examples show that convexity is essential in this characterization (see **[13]**). The authors of **[7]** present a pair of non-congruent non-convex polygons with equal covariogram. Bianchi (see **[14]**) found counterexamples to the covariogram conjecture in dimensions greater than or equal to 4, and a positive answer for three-dimensional polytopes (see [**16**]).

The general three-dimensional case is still open. For dimensions greater than or equal to 3, most convex bodies, in the sense of Baire category, are determined by their covariogram; this was proved by Goodey, Schneider and Weil (see [**15**]).

Determine a random line of direction $u \in S^{n-1}$. Given a convex domain D we define its breadth function b(D, u) as the Lebesgue measure of the projection of D on the hyperplane with direction $u^{\perp}$ ($u^{\perp}$ is the orthogonal complement to u).

A line parallel to u and intersecting D is intersected with hyperplanes with direction $u^{\perp}$. Let fix one of these hyperplanes and denote by z the corresponding intersection point. Thus we have one-to-one correspondence between lines parallel to u and intersecting D and all points z. Projecting D on the fixed hyperplane and assume that z has uniform distribution in the projection of D in the fixed hyperplane with direction $u^{\perp}$. We have defined the notion of line parallel to u.

Denote by $F_D(u,x)$ orientation-dependent chord length distribution function. Determination of a convex body D by these distributions, for all directions, is equivalent to the determination by its covariogram. Matheron (see [**8**] and [**9**]) obtained relationship between $F_D(u,x)$ and covariogram:

Let $u \in S^{n-1}$ and $x > 0$ such that the set $D \cap (D + xu)$ contains interior points: In this case $C(D,xu)$ is differentiable in x and the following equation

$$-\frac{\partial C(D,xu)}{\partial x} = L_{n-1}(y \epsilon\ u^{\perp} :\ L_1(D \cap l_u + y) \geq x) \qquad (1)$$

Where $l_u + y$ denotes the line parallel to u through y. This formula allows some interpretation of the covariogram problem. The right hand side of (1) gives the distribution of the lengths of the chords of D which are parallel to u.

If we rewrite (1) by means of orientation-dependent chord length distribution, we get

$$-\frac{\partial C(D,xu)}{\partial x} = (1 - F_D(u,x)) \cdot\ b(D,u)$$

and at the point $x = 0$ there exists right derivative.

Therefore, investigation of convex bodies is equivalent to investigation of their covariograms. If we have covariogram for a convex body, then we can investigate properties of the convex body by explicit form of the covariogram. Obtaining the explicit form of the covariogram for any convex body is very difficult problem, but we can obtain covariograms for a subclass of convex bodies. These forms help us to solve many probabilistic problems, in particular calculate explicit forms of chord length distribution functions $F_D(x)$ and $F_D(u,x)$ (see, for instance, [**17**]).

There exist one-to-one mapping between the set of all bounded convex bodies D and their $F_D(u,x)$ chord length distribution functions up to translations and reflection.

The explicit form of covariogram for subclasses of convex bodies allows us to obtain new properties of covariogram and generalized these results we can answer the question what functions will be covariograms of convex bodies. Similar problems have been suggested by Bianchi (see **[16]** and [**18**]).

In the case n=2 we can use the generalized Pleijel identity (see **[20]**) to determine covariograms and oriented-dependent chord length distributions for subclasses of convex bodies.

In the generalized Pleijel identity integration is over arbitrary locally finite, bundleless measure in the space **G** (a measure is called bundleless, if the measure of a bundle of lines through the point P is equal to zero for any point P in the plane): In the space G there exists a unique (up to a constant) measure, which is invariant with respect to translations and rotations: Denote by dg the element of the invariant measure. It is known that

$$dg = dpd\varphi,$$

where dp is the one dimensional Lebesgue measure, and dφ is a uniform measure in S^1. For a convex body D we set

$$[D] = \{g \in G, g \cap D \neq \emptyset\}.$$

It is obtained in [**20**] that

$$\mathbf{b}(D,\mathbf{u})[1 - F_D(u^{\perp},\mathbf{x})] = \frac{1}{2}\int_{[D]} \delta(|\chi(g)| - x)|\chi(g)||\sin(\varphi - u)|\, dg -$$

$$-\frac{1}{2}\int_{[D]} \delta'(|\chi(g)| - x)|\chi(g)|^2|\sin(\varphi - u)|\, cot\alpha_1 cot\alpha_2 dg$$

where α_1 and α_2 are the angles between the boundary of D and the line g, at the endpoints of $\chi(g) = g \cap D$ which lie in one half-plane with respect to the line g and inside of D and δ (x) is the Dirac delta-function.

In the paper **[17]** is proved that for any finite subset A from S^1, there are two non-congruent domains for which orientation-dependent chord length distribution functions coincide for any direction from A. Moreover, in **[17]** explicit forms for covariogram and orientation-dependent chord length distribution function $F_\Delta(u,x)$ for arbitrary triangle are obtained. Finally, if we have the values of $F_\Delta(u,x)$ for everywhere dense set from S^1 then we can uniquely recognized the triangle with respect to translations and reflections (see [**17**] and [**22**] and [**23**]). Thus, investigating covariograms of convex bodies we investigate the geometric properties of them.

Find the explicit form of covariogram for subclass of convex domains: Using the explicit form of covariogram and Materon's formula find $F_D(u,x)$ for the corresponding subclass of convex bodies. Construct algorithms to reconstruct convex body by its covariogram for finite number of directions (see **[18]**) (the same problem for chord length distribution function $F_D(u,x)$ (see **[19]**) in finite number of directions has negative solution (see **[17]**):

The explicit forms for Covariogram of a triangle and for $F_D(u,x)$ follows that their can be written in the form (see **[17]**):

$$C(D, xu) = L_2(D)\left(1 - \frac{x}{t_{max}(u)}\right)^2$$

$$F_D(u,x) = \frac{x}{t_{max}(u)}$$

where $t_{max}(u)$ is the maximal chord length in direction u.

In the last 3 years our group has obtained important results to calculate explicit forms of chord length distribution functions $F_D(x)$ and $F_D(u,x)$ for different convex bodies. In particular, if D is a lens (this problem has important applications in crystallography). An algorithm for calculation of values of $F_D(x)$ for any bounded convex polygon is constructed. The program for effective implementation of this algorithm is constructed. For any triangle explicit forms of $F_D(x)$ and $F_D(u, x)$ are obtained (see [**17**], [**19**]-[**24**].

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Распознавание выпуклых тел ковариограммами

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Аннотация. Данная работа содержит анализ основных результатов Ереванской исследовательской группы в томографии планарных ограниченных выпуклых доменов. Приложения этих проблем известны и в геометрической томографии и в компьютерной томографии. Сложные геометрические фигуры происходят во многих областях науки. Их анализ требует создания математических моделей и разработки специальных математических инструментов. Соответствующую область математического исследования вызывают стохастической геометрией. Среди более популярных приложений Стереология и томография. Методы анализа формы основываются на анализе объектов как числа, т.е. как подмножества плоскости. Для этих наборов геометрические характеристики рассматривают, которые независимы от позиции и ориентации чисел (следовательно, они совпадают для конгруэнтных фигур). Классические примеры – область и периметр числа. В последнем немецком математике столетия В. Бляшке сформулировал проблему исследования ограниченных выпуклых доменов в плоскости, используя вероятностные методы. В частности проблема распознавания ограниченных выпуклых доменов распределением длины хорды.

Ключевые слова: Распределение длины хорды; ковариограмма; Классификация 2010 предметов математики выпуклого тела: 60D05; 52A22; 53C65.