Impact Factor:

ISRA (India)	= 6.317	SIS (USA)	= 0.912	ICV (Poland)	= 6.630
ISI (Dubai, UAE)	= 1.582	РИНЦ (Russia)	= 0.126	PIF (India)	= 1.940
GIF (Australia)	= 0.564	ESJI (KZ)	= 9.035	IBI (India)	= 4.260
JIF	= 1.500	SJIF (Morocco)	= 7.184	OAJI (USA)	= 0.350

QR – Issue QR – Article

SOI: 1.1/TAS DOI: 10.15863/TAS

International Scientific Journal

Theoretical & Applied Science

p-ISSN: 2308-4944 (print) **e-ISSN:** 2409-0085 (online)

Year: 2021 **Issue:** 07 **Volume:** 99

Published: 19.07.2021 http://T-Science.org

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SOLVABILITY OF WEIGHTED INITIAL PROBLEM FOR THE HIGH-ORDER NON-LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

***Abstract**: The paper dwells on establishing the becessary and sufficient conditions of solvability of weighted problem for the high-order non-linear functional-differential equations.*

***Key words**: Nonlinear singular differential equation with a delay, the Caucly weighted problem, solvability, 2010 mathematics Subject Classification 34k05.*

***Language**: English*

***Citation**: Sokhadze, Z. P., & Shalamberidze, M. M. (2021). Solvability of weighted initial problem for the high-order non-linear functional-differential equations. *ISJ Theoretical & Applied Science, 07 (99),* 108-110.

***Soi**: http://s-o-i.org/1.1/TAS-07-99-22 ***Doi**: crossref https://dx.doi.org/10.15863/TAS.2021.07.99.22

***Scopus ASCC**: 2601.*

Introduction

Let us consider an nth-order non-linear functional-differential equation within the finite interval $[a,b]$:

$$u^{(n)}(t) = f\left(t, u\left(\tau_1(t)\right), \dots, u^{(n-1)}\left(\tau_n(t)\right)\right) \quad (1)$$

With the weighted initial conditions

$$\lim_{t\to a}\left(\rho(t)u^{(i-1)}(t)\right) = 0 \quad (i=1,\dots,n) \quad (2),$$

where the function $f:]a,b[\times \mathbb{R}^{u} \to \mathbb{R}$ satisfies local Caratheodory conditions, $\tau_i:]a,b[\to]a,b[\quad (i=1,\dots,n)$ are the measurable functions, but $\rho:]a,b[\to]0,+\infty[$ is a nonincreasing function.

Let us note that

$$f^*(t,x) = \max\left\{\left|f(t,x_1,\dots,x_n)\right| : \sum_{i=1}^{n}|x_i| \le x\right\},$$

when $a \le t \le b, \quad x > 0$

From here on we assume that

$$\int_t^{\ell} f^*(s,x)\,ds < +\infty, \text{ when } a < t < b, \quad x > 0$$

Note that the equation (1) has singularity towards atemporary variable at the point t=α, if

$$\int_a^b f^*(s,x)\,ds = +\infty, \text{ when x>0}$$

This singularity is called strong, if

$$\int_a^b s^{\mu} f^*(s,x)\,ds = +\infty, \text{ when x>0 and } \mu > 0$$

The positions we established on solvability of problem (1), (2) includes the case, when the equation (1) has strong singularity towards a temporary variable at the point t=α.

Along with the equation, we shall consider an auxiliary differential equation

$$u^{(n)}(t) = \lambda(t)\, f\left(t, u\left(\tau_1(t)\right), \dots, u^{(n-1)}\left(\tau_n(t)\right)\right), \quad (3)$$

where $\lambda: [a,b] \to [0,1]$, is any continuous function.

Theorem 1. (Principle of a priori restriction). Assume that there can be found such a continuous function $\delta:[a,b]\to[0,+\infty[$ that $\delta(a)=0$, and for each continuous function $\lambda:[a,b]\to[0,1]$, any solution to the problem (2), (3) satisfies the inequation

$\int(t)\left|u^{(n-1)}(t)\right|\le\delta(t)$, when $a<t\le b$

Then the problem (1), (2) has at least one solution.

This Theorem allows us for establishing effective and, to some extent, unimprovable conditions. However, to attain this goal, we should need to additionally study the differential inequation

$$\left|u^{(n)}(t)\right|\le\sum_{k=1}^{n}h_k(t)\left|u^{(k-1)}(\tau_k(t))\right|+h_0(t) \qquad (4)$$

(2) with initial conditions.

The coefficients $h_k:]a,b[\to[0,\infty[\quad (k=0,...,n)$ of inequation (4) are the integratable functions for whatever small positive ε, within the interval $[a+\varepsilon,b]$, but at the point ɑ, they generally have non-integratable singularity.

Let us introduce the operator

$$q_m(h_1,...,h_n)(t)=\sum_{k=1}^{m}\frac{h_k(t)}{(n-k-1)!}\left|\int_t^{\tau_k(t)}\frac{(S-G)}{\rho(S)}dS\right|+\sum_{k=m+1}^{n}\frac{(\tau_k(t)-a)^{n-k}}{(n-k)!\rho(\tau_k(t))}$$

for each $m\in\{1,...,n-1\}$, where $a<t<b$.

Lemma 1. Assume that there exists $m\in\{1,...,n-1\}$, $\delta_0\in]0,1[$ and a nonincreasing continuous function $\gamma:[a,b]\to]0,+\infty[$ is such that

$$\exp\left(\sum_{k=1}^{m}\frac{(x-a)^{n-k}}{(n-k)!}h_k(x)dx\right)\le\frac{\rho(s)}{\rho(t)},\ \text{when } a<S<t<b \qquad (5)$$

and

$$\gamma(t)\int_a^t\frac{\rho(s)q_m(h_1,...,h_n)(s)}{\gamma(\tau_0(s))}\le\delta_0,\ \text{when } a<t\le b \qquad (6)$$

where

$\tau_0(t)=\max\{t,\tau_1(t),...,\tau_n(t)\}$

And besides, if

$$\int_a^b\rho(s)h_0(s)ds<+\infty$$

then a fair assessment for each solution to the problem (4),(2) is

$$\rho(t)\left|u^{(n-1)}(t)\right|\le\frac{\gamma(a)}{(1-\delta)\gamma(b)}\int_a^t\rho(s)h_0(s)ds,$$

when $a<t\le b$

Based on Theorem 1 and Lemma 1, there is proved

Theorem 2. Assume that within the the area $]a,b[\times\mathbb{R}^n$ there is ended the inequation

$$\left|f(t,x_1,...,x_n)\right|\le\sum_{k=1}^{n}h_k(t)\left|x_k\right|+h_0(t) \quad (7),$$

where $h_k:]a,b[\to]0,\infty[\quad(k=0,...,n)$ are integratable functions for whatever small positive ε, within the interval $[a+\varepsilon,b]$, but h_0 is a weighted integratable function. Assume that in addition to this, there can be found such $\delta_0\in]0,1[$ and $m\in\{1,...,n-1\}$ that the inequations (5) and (6) are satisfied. Then, the problem (1), (2) has at least one solution.

Note 1. In this Theorem, te condition $\delta\in]0,1[$ is unimprovable $\delta_0=1$, and it cannot be replaced by the condition $\delta\ =1$.

Theorem 3. Assume that $\tau_k(t)\le t$, when $a<t<b$ (k=1,..,n) (8) and within the area $]a,b[\times\mathbb{R}^n$ the inequation (8) is satisfied, where $h_k:]a,b[\to]0,+\infty[$ (u=1,..,n) is within the interval $[a+\varepsilon,b]$ are the integratable functions for whatever small positive ε, but h_0 is a ρ weighted integratable function. Assume that in addition to this, there can be found such $m\in\{1,...,n-1\}$ that the inequation (5) is satisfied and

$$\int_a^b\rho(s)q_m(h_1,...,h_n)(S)ds<+\infty \qquad (9)$$

Then, the problem (1), (2) has at least one solution.

Note 2. It is obvious that because of Note 1, if the condition (8) is violated, the condition (6) cannot be replaced by the condition (9).

The issue of the existence singularity of a solution to problem (1), (2), when ($\tau_i(t) \equiv t$ (i=1,...,n) has been studied in the works [1-5]. The weighted problem with strong singularity for the system of non-linear differential equations has been studied in the works [6-8].

For some special cases of (1), (2) of the problem (1), (2), there are addressed various engineering-technological topical problems [9].

Acknowledgement: The work was fulfilled with the financial support of Shota Rustaveli National Science Foundation, Grant FR № 217386.

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