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On (n, m)-Metrically Equivalent Operators

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Abstract: In this paper, we introduce the class of (n,m)-metrically equivalent operators which is a generilazation of metrically equivalent operators and n-metrically equivalent operators. We then look at some properties of this class and its relation to some higher classes like quasi-isometries and the (n,m)-class (Q) operators. we also look at the relationship between this class and other equivalence relations like metrically equivalent and n-metrically equivalent operators.

Keywords: (n,m)-metrically equivalent, n-metrically equivalent, metrically equivalent, (n,m)-class(Q), normal and n-normal operators.

1. Introduction

Definition 1.1. *Two operators $S \in B(H)$ and $T \in B(H)$ are said to be (n, m)-metrically equivalent denoted by $S \sim_{\text{(n,m)-m}} T$, provided $(S^m)^*S^n = (T^m)^*T^n$ for all $n, m \in \mathbb{R}$.*

Definition 1.2 ([5]). *Two operators $S \in B(H)$ and $T \in B(H)$ are said to be (n, m)-metrically equivalent denoted by $S \sim_{\text{n-m}} T$, provided $(S)^*S^n = (T)^*T^n$ for all $n \in \mathbb{R}$.*

Definition 1.3 ([3]). *Two operators $S \in B(H)$ and $T \in B(H)$ are said to be metrically equivalent denoted by $S \sim_{\text{m}} T$, provided $S^*S = T^*T$.*

Definition 1.4 ([1]). *Two operators $T \in B(H)$ is said to be (n, m)-class(Q) if $T^{*2m}T^{2n} = (T^{m*}T^n)^2$ for non negative integers n, m.*

2. Main Results

Theorem 2.1. *If S is an (n, m)-normal operator and $T \in B(H)$ is unitarily equivalent to S, then T is an (n, m)-normal.*

Proof. Since $T = U^*SU$ with U being unitary and S (n, m)-normal, we have

$$
\begin{aligned}
(T^m)^*T^n &= U^*(S^m)^*S^nU \\
(T^m)^*T^n &= U^*(S^m)^*UU^*S^nU \\
&= U^*(S^m)^*S^nU \\
&= U^*S^n(S^m)^*U
\end{aligned}
$$

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$$= T^n U^*(S^m)^* U$$
$$= T^n U^* U (T^m)^*$$
$$= T^n (T^m)^*$$

which proves the claim. □

Corollary 2.2. *An operator $T \in B(H)$ is (n, m)-normal if and only if T and T^* are (n, m)-metrically equivalent.*

Proof. The proof follows from Theorem 2.1. □

Proposition 2.3. *Let S and T be (n, m)-metrically equivalent, then S^* and T^* are co-(n, m)-metrically equivalent.*

Proof. Since S and T are (n,m)-metrically equivalent, we have,

$$(S^m)^* S^n = (T^m)^* T^n, \quad \text{taking adjoints on both sides we obtain;}$$
$$= ((S^m)^* S^n)^*$$
$$= ((T^m)^* T^n)^*$$
$$= ((S^m)^*)^* (S^n)^*$$
$$= ((T^m)^*)^* (T^n)^*$$
$$= S^m (S^n)^*$$
$$= T^m (T^n)^*$$

hence S^* and T^* are co-(n, m)-metrically equivalent. □

Theorem 2.4. *Let $T_{\alpha_i} \dots T_{\alpha_r}$ and $S_{\alpha_i} \dots S_{\alpha_r}$ be (n, m)-metrically equivalent operators. Then $T_{\alpha_i} \oplus \dots \oplus T_{\alpha_r}$ and $S_{\alpha_i} \oplus \dots \oplus S_{\alpha_r}$ are (n,m)-metrically equivalent.*

Proof. Since $T_{\alpha_i} \dots T_{\alpha_r}$ and $S_{\alpha_i} \dots S_{\alpha_r}$ are (n, m)-metrically equivalent operators, we have;

$$= ((T_{\alpha_i} \oplus \dots \oplus T_{\alpha_r})^m)^* (T_{\alpha_i} \oplus \dots \oplus T_{\alpha_r})^n$$
$$= ((S_{\alpha_i} \oplus \dots \oplus S_{\alpha_r})^m)^* (S_{\alpha_i} \oplus \dots \oplus S_{\alpha_r})^n$$
$$= (T^m_{\alpha_i} \oplus \dots \oplus T^m_{\alpha_r})^* (T^n_{\alpha_i} \oplus \dots \oplus T^n_{\alpha_r})$$
$$= (S^m_{\alpha_i} \oplus \dots \oplus S^m_{\alpha_r})^* (S^n_{\alpha_i} \oplus \dots \oplus S^n_{\alpha_r})$$
$$= (T^{m*}_{\alpha_i} \oplus \dots \oplus T^{m*}_{\alpha_r})(T^n_{\alpha_i} \oplus \dots \oplus T^n_{\alpha_r})$$
$$= (S^{m*}_{\alpha_i} \oplus \dots \oplus S^{m*}_{\alpha_r})(S^n_{\alpha_i} \oplus \dots \oplus S^n_{\alpha_r})$$
$$= T^{m*}_{\alpha_i} T^n_{\alpha_i} \oplus \dots \oplus T^{m*}_{\alpha_r} T^n_{\alpha_r}$$
$$= S^{m*}_{\alpha_i} S^n_{\alpha_i} \oplus \dots \oplus S^{m*}_{\alpha_r} S^n_{\alpha_r}$$
$$= (T^{m*}_{\alpha_i} \oplus \dots \oplus T^{m*}_{\alpha_r})(T^n_{\alpha_i} \oplus \dots \oplus T^n_{\alpha_r})$$
$$= (S^{m*}_{\alpha_i} \oplus \dots \oplus S^{m*}_{\alpha_r})(S^n_{\alpha_i} \oplus \dots \oplus S^n_{\alpha_r})$$
$$= (T^m_{\alpha_i} \oplus \dots \oplus T^m_{\alpha_r})^* (T^n_{\alpha_i} \oplus \dots \oplus T^n_{\alpha_r})$$
$$= (S^m_{\alpha_i} \oplus \dots \oplus S^m_{\alpha_r})^* (S^n_{\alpha_i} \oplus \dots \oplus S^n_{\alpha_r})$$
$$= ((T_{\alpha_i} \oplus \dots \oplus T_{\alpha_r})^m)^* (T_{\alpha_i} \oplus \dots \oplus T_{\alpha_r})^n$$

$$= ((S_{\alpha_i} \oplus \cdots \oplus S_{\alpha_r})^m)^*(S_{\alpha_i} \oplus \cdots \oplus S_{\alpha_r})^n$$

hence $T_{\alpha_i} \oplus \cdots \oplus T_{\alpha_r}$ and $S_{\alpha_i} \oplus \cdots \oplus S_{\alpha_r}$ are (n, m)-metrically equivalent operators. □

Theorem 2.5. *Let $T_{\alpha_i} \ldots T_{\alpha_r}$ and $S_{\alpha_i} \ldots S_{\alpha_r}$ be (n, m)-metrically equivalent operators. Then $T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r}$ and $S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r}$ are (n, m)-metrically equivalent.*

Proof. Let $x_{\alpha_i} \ldots x_{\alpha_r} \in H$, it follows that;

$$= ((T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r})^m)^*(T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r})^n(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= ((S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r})^m)^*(S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r})^n$$
$$= (T^m_{\alpha_i} \otimes \cdots \otimes T^m_{\alpha_r})^*(T^n_{\alpha_i} \otimes \cdots \otimes T^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (S^m_{\alpha_i} \otimes \cdots \otimes S^m_{\alpha_r})^*(S^n_{\alpha_i} \otimes \cdots \otimes S^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (T^{m*}_{\alpha_i} \otimes \cdots \otimes T^{m*}_{\alpha_r})(T^n_{\alpha_i} \otimes \cdots \otimes T^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (S^{m*}_{\alpha_i} \otimes \cdots \otimes S^{m*}_{\alpha_r})(S^n_{\alpha_i} \otimes \cdots \otimes S^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= T^{m*}_{\alpha_i} T^n_{\alpha_i} \otimes \cdots \otimes T^{m*}_{\alpha_r} T^n_{\alpha_r}(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= S^{m*}_{\alpha_i} S^n_{\alpha_i} \otimes \cdots \otimes S^{m*}_{\alpha_r} S^n_{\alpha_r}(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (T^{m*}_{\alpha_i} \otimes \cdots \otimes T^{m*}_{\alpha_r})(T^n_{\alpha_i} \otimes \cdots \otimes T^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (S^{m*}_{\alpha_i} \otimes \cdots \otimes S^{m*}_{\alpha_r})(S^n_{\alpha_i} \otimes \cdots \otimes S^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (T^m_{\alpha_i} \otimes \cdots \otimes T^m_{\alpha_r})^*(T^n_{\alpha_i} \otimes \cdots \otimes T^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= (S^m_{\alpha_i} \otimes \cdots \otimes S^m_{\alpha_r})^*(S^n_{\alpha_i} \otimes \cdots \otimes S^n_{\alpha_r})(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= ((T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r})^m)^*(T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r})^n(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$
$$= ((S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r})^m)^*(S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r})^n(x_{\alpha_i} \otimes \cdots \otimes x_{\alpha_r})$$

hence $T_{\alpha_i} \otimes \cdots \otimes T_{\alpha_r}$ and $S_{\alpha_i} \otimes \cdots \otimes S_{\alpha_r}$ are (n, m)-metrically equivalent operators. □

Theorem 2.6. *If S and T are (n, m)-metrically equivalent operators then they are (n, m)-power class (Q).*

Proof. Since S and T are (n, m)-metrically equivalent;

$$S^{*m}S^n = T^{*m}T^n \tag{1}$$

post -multiplying both sides of (1) by S^n and T^n respectively;

$$S^{*m}S^nS^n = T^{*m}T^nT^n \tag{2}$$

$S^{*m}S^{2n} = T^{*m}T^{2n}$ pre-multiplying both sides of (2) by S^{*m} and T^{*m} respectively;

$$S^{*m}S^{*m}S^{2n} = T^{*m}T^{*m}T^{2n}$$
$$S^{*2m}S^{2n} = T^{*2m}T^{2n}$$
$$S^{*2m}S^{2n} = S^{*m}S^{*m}S^nS^n$$
$$= (S^{*m}S^n)^2$$

$$= (T^{*m}T^n)^2$$
$$= T^{*m}T^{*m}T^nT^n$$
$$= T^{*2m}T^{2n}.$$

□

Theorem 2.7. *If S and T are (2,2)-metrically equivalent operators, then they are metrically equivalent provided they are quasi-isometries.*

Proof. The proof is trivial and follows from the fact that if S and T are (2,2)-metrically equivalent, then we have

$$S^{*2}S^2 = T^{*2}T^2 \tag{3}$$

since S and T are quasi-isometries; we have $S^*S = S^{*2}S^2$ and $T^*T = T^{*2}T^2$, hence (3) gives us $S^*S = T^*T$. □

Theorem 2.8. *If S and T are (3,3)-metrically equivalent operators and S is (2,3)-quasinormal, then T is (2,3)-quasinormal.*

Proof.

$$\begin{aligned}(S^3)^*S^3 &= U(T^3)^*T^3U^* \\ &= (S^3)^*SS^2 \\ &= S^2(S^3)^*S \\ &= U(T^3)^*T^3U^* \\ &= (T^3)^*T^3 \\ &= (T^{3*})TT^2 \\ &= T^2T^{3*}T \\ &= T^{3*}TT^2\end{aligned}$$

□

Remark 2.9. *In the following proposition, we provide a condition under which (2,1)-metrically equivalent operators implies metric equivalence relation.*

Proposition 2.10. *If S and T are (2,1)-metrically equivalent operators , then they are metrically equivalent provided they are idempotent.*

Proof. Since S and T are (2,1)-metrically equivalent, we have $S^*S^2 = T^*T^2$, since S and T are idempotent we have $S^2 = S$ and $T^2 = T$, this implies $S^*S^2 = T^*T^2 \Leftrightarrow S^*S = T^*T$ as required. □

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