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Existence of solutions for hybrid Caputo fractional $q-$ differential equations

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ABSTRACT

This paper studies the existence of solutions for Caputo fractional hybrid q-differential equations. The existence obtained by using the Leray-Schauder's alternative, while the uniqueness of solutions is established by means of Banach's contraction mapping principle. An illustrative examples are also included.

1. Introduction

In this work, we discuss the existence and uniqueness of solutions for the following fractional $q-$differential hybrid equations :

$$\begin{cases} D_q^{\alpha}\left(\frac{x(t)}{g(t,x(t))}\right) = f\left(t, x\left(t\right)\right), t \in J = \left[0,1\right], 0 < q < 1, \\ \\ x\left(0\right) = 0,\ x\left(1\right) - \theta = 0, \end{cases} \tag{1}$$

where D_q^{α} denote the Caputo fractional derivative, with $1 < \alpha \leq 2, \theta \in \mathbb{R}$, $f \in C\left(J \times \mathbb{R}, \mathbb{R}\right)$, $g \in C\left(J \times \mathbb{R}, \mathbb{R} - \{0\}\right)$. In recent years, boundary value problems of fractional $q-$ differential equations involving a variety of boundary conditions have been investigated by several researchers, see for example [1, 9, 14, 15, 17]. By using fixed point theory, many researchers have established existence results for some $q-$differential equations. For more details, we refer the reader to [14, 16, 20, 22, 23] and the references therein. Fractional hybrid differential equations have recently been studied by several researchers. For

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some earlier work on the topic, we refer to [3, 5, 6, 7, 8, 12], whereas some recent work on the existence theory of fractional hybrid differential equations can be found in [10, 11, 13, 19, 21]. Recently, many researchers have studied the existence of solutions for some fractional ($q-$fractional) hybrid difference equations. For some recent work on fractional differential equations, we refer to [1, 17, 20, 23] and the references therein.

The rest of this paper is organized as follows : In section 2, basic definitions and notations are given. Section 3 is devoted to establish an existence and uniqueness result for the $q-$fractional hybrid problem (1). The first main result is based on Leray Schauder alternative and the second on Banach contraction principle. In section 4, an illustrative examples are discussed.

2. Preliminaries

We give some necessary definitions and mathematical preliminaries of fractional $q-$calculus. More details, one can consult [2, 4, 18].

For a real parameter $q \in (0,1)$, a $q-$real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The $q-$analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}$ is

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \prod_{j=0}^{n-1} \left(a - bq^j\right),\ n \in \mathbb{N}^*,\ a, b \in \mathbb{R}.$$

More generally, if $\beta \in \mathbb{R}$, then

$$(a-b)^{(\beta)} = a^\beta \prod_{i=0}^{\infty} \frac{a - bq^i}{a - bq^{\beta+i}}.$$

It is easy to see that $[a(y-z)]^{(\beta)} = a^\beta (y-z)^{(\beta)}$. And note that if $b = 0$ then $a^{(\beta)} = a^\beta$.

The $q-$gamma function is defined by

$$\Gamma_q(v) = \frac{(1-q)^{(v-1)}}{(1-q)^{v-1}},\ v \in \mathbb{R}\backslash \{0, -1, -2, ...\},\ 0 < q < 1,$$

and satisfies $\Gamma_q(v+1) = [v]_q \Gamma_q(v)$.

The $q-$derivative of a function f is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(1-q)x},\ x \neq 0,\ (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and the $q-$derivatives of higher order by

$$D_q^0 f = f,\ D_q^n f = D_q \left(D_q^{n-1} f\right),\ n \in \mathbb{N}^*.$$

The $q-$integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n,\ x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its $q-$integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined as

$$I_q^0 f(x) = f(x), \ I_q^n f(x) = I_q\left(I_q^{n-1} f\right)(x), \ n \in \mathbb{N}^*.$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$D_q I_q f(x) = f(x),$$

Definition 1 *[2]. Let $\alpha \geq 0$ and f be a function defined on $[0,T]$. The fractional $q-$integral of the Riemann-Liouville type is given by $I_q^0 f(t) = f(t)$ and*

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(t)\, d_q t, \ \alpha > 0, \ t \in [0,T].$$

Definition 2 *[4]. The Caputo fractional $q-$derivative of order $\alpha \geq 0$ is defined by*

$$D_q^\alpha f(t) = I_q^{m-\alpha} D_q^m f(t), \ \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

For more details on $q-$integral and fractional $q-$integral, we refer the reader to [18].

Lemma 3 *[4]. Let $\alpha, \beta \geq 0$ and f be a function defined in $[0,T]$. Then the following formulas hold :*

1. $I_q^\alpha I_q^\beta f(t) = I_q^{\alpha+\beta} f(t);$

2. $D_q^\alpha I_q^\alpha f(t) = f(t).$

Lemma 4 *[4]. Let $\alpha > 0$ and σ be a positive integer. Then the following equality holds :*

$$I_q^\alpha D_q^\sigma f(t) = D_q^\sigma I_q^\alpha f(t) - \sum_{j=0}^{\sigma-1} \frac{t^{\alpha-\sigma+j}}{\Gamma_q(\alpha+j-\sigma+1)} D_q^j f(0).$$

Lemma 5 *[4]. Let $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$ and $a < t$, then the following is valid :*

$$I_q^\alpha D_q^\alpha f(t) = f(t) - \sum_{j=0}^{n-1} \frac{t^j}{\Gamma_q(j+1)} D_q^j f(0).$$

Lemma 6 *Let $g \in C((J, \mathbb{R} - \{0\})$ and $\varphi \in C(J, \mathbb{R})$. Then, the solution of the equation*

$$ {}^c D_q^\alpha \left(\frac{x(t)}{g(t, x(t))} \right) = \varphi(t), \ t \in J, \ 0 < q < 1, \ 1 < \alpha \leq 2, \tag{2}$$

subject to the boundary conditions

$$x(0) = 0, \ x(1) - \theta = 0, \tag{3}$$

given by :

$$\begin{aligned} x(t) &= g(t, x(t)) \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, d_q s \right. \\ &\quad \left. + t \left[\theta - \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, d_q s \right] \right). \end{aligned} \tag{4}$$

Proof. Applying the operator I_q^α on the equation $D_q^\alpha\left(\frac{x(t)}{g(t,x(t))}\right)=\varphi(t)$ and using (3), we get

$$x(t)=g(t,x(t))\left(I_q^\alpha\varphi(t)+c_0+c_1t\right), \tag{5}$$

where c_0, c_1 are arbitrary constants.
Using the condition (3), we can write $c_0=0$ and

$$c_1=\theta-I_q^\alpha\varphi(1).$$

Substituting the value of c_0 and c_1 in (5), we obtain the desired quantity in Lemma 6.

3. Main Results

We denote by $X=C(J,\mathbb{R})$ the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by :

$$\|x\|=\sup\{|x(t)|:t\in J\}.$$

In view of Lemma 6, we define the operator $O:X\to X$,

$$\begin{aligned} Ox(t) &= g(t,x(t))\left(\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s))\,d_qs\right. \\ &\quad \left.+t\left[\theta-\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s))\,d_qs\right]\right). \end{aligned} \tag{6}$$

We impose the following hypotheses :

(H_1) : There exists nonnegative constant k, such that for all $t\in J$ and $x\in\mathbb{R}$, we have :

$$|f(t,x)-f(t,y)|\le k|x-y|.$$

(H_2) : There exists nonnegative constant N such that for each $t\in J$ and $x\in\mathbb{R}$, $|g(t,x)|\le N$.

(H_3) : There exist real constants $\rho>0$ and $\sigma\ge 0$ such that, $\forall\in x\in\mathbb{R}$ we have

$$|f(t,x)|\le\rho+\sigma|x|.$$

In our first result, we discuss the existence of solutions for the problem (1) by means of Leray-Schauder alternative [11].

Lemma 7 *(Leray-Schauder alternative). Let $F:E\to E$ be a completely continuous operator (i.e.,a map that restricted to any bounded set in E is compact). Let*

$$\Theta(F)=\{u\in E:u=\lambda F(u)\ \textit{for some}\ 0<\lambda<1\}.$$

Then either the set $\Theta(F)$ is unbounded, or F has at least one fixed point.

Theorem 8 *Let $f:J\times\mathbb{R}\to\mathbb{R}$ be continuous functions satisfying the conditions (H_2) and (H_3). In addition it is assumed that*

$$2N\sigma<\Gamma_q(\alpha+1), \tag{7}$$

Then the problem (1) has at least one solution on J.

Proof. In the first step, we show that the operator $O : X \to X$ is completely continuous. By continuity of the functions $f,\ g$, it follows that the operator T is continuous.

Let $\Theta \subset X$ be bounded. Then, there exist positive constants L such that
$|f(t, x(t))| \leq L, \forall x \in \Omega.$

Then for any $x \in \Theta$, we have

$$\begin{aligned} |Ox(t)| &\leq |g(t, x(t))| \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s))| \, d_q s \right. \\ &\quad \left. + t \left[|\theta| + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s))| \, d_q s \right] \right) \\ &\leq N \left[\frac{L}{\Gamma_q(\alpha+1)} + \theta + \frac{L}{\Gamma_q(\alpha+1)} \right], \end{aligned}$$

which implies that
$$\|O(x)\| \leq N \left[\frac{2L}{\Gamma_q(\alpha+1)} + \theta \right].$$

From the above inequalitie, it follows that the operator O is uniformly bounded.

Next, we show that O is equicontinuous. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned} & |O(x)(t_2) - O(x)(t_1)| \\ \leq & \frac{NL}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)} \right] d_q s \\ & + \frac{N_1 M_1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} \left[(t_2-qs)^{(\alpha-1)} \right] d_q s + |t_2 - t_1| \left[N\theta + \frac{NL}{\Gamma_q(\alpha+1)} \right] \\ \leq & \frac{NL}{\Gamma_q(\alpha+1)} \left| t_2{}^{(\alpha)} - t_1{}^{(\alpha)} \right| + |t_2 - t_1| \left[N\theta + \frac{NL}{\Gamma_q(\alpha+1)} \right]. \end{aligned}$$

we can obtain which is independent of x and tends to zero as $t_2 - t_1 \to 0$. This implies that the operator O is equicontinuous. Thus, by the above findings, the operator O is completely continuous.

Finally, it will be verified that the set $\Theta = \{x \in X, x = \lambda O(x), 0 < \lambda < 1\}$ is bounded. Let $x \in \Theta$, then, $x = \lambda O(x)$. For each $t \in J$, we can write
$x(t) = \lambda O(x)(t).$

Then
$$\|x\| \leq \frac{2N}{\Gamma_q(\alpha+1)} (\rho + \sigma \|x\|) + N\theta,$$

which imply that
$$\|x\| \leq \frac{2N\rho}{\Gamma_q(\alpha+1)} + \frac{2N\sigma}{\Gamma_q(\alpha+1)} \|x\| + N\theta.$$

Consequently,
$$\|x\| \leq \frac{2N\rho - N\theta\Gamma_q(\alpha+1)}{\Gamma_q(\alpha+1) - 2N\sigma},$$

This shows that the set Θ is bounded. Hence, by Lemma 7, the operator O has at least one fixed point. Hence, problem (1) has at least one solution. The proof is complete.

Now, we are in a position to present the existence and uniqueness of solutions for the problem (1). This result is based on Banach's contraction mapping principle.

Theorem 9 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the conditions (H_1) and (H_2).*

If
$$2Nk < \Gamma_q(\alpha+1), \tag{8}$$

then the problem (1) has a unique solution on J.

Proof. Define $\sup_{t\in J} |f(t,0)| = M < \infty$, such that
$$r \geq \frac{NM\Gamma_q(\alpha+1) + 2N\theta}{\Gamma_q(\alpha+1) - 2Nk}.$$

We first show that $TB_r \subset B_r$; where $B_r = \{x \in X : \|x\| < r\}$. For $x \in B_r$, we find the following estimates based on the assumptions (H_1), we observe that

$$\begin{aligned} |f(t,x(t))| &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| \\ &\leq k\|x\| + M \leq kr + M. \end{aligned} \tag{9}$$

Hence, we get

$$\begin{aligned} |Ox(t)| \leq\ & |g(t,x(t))| \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,x(s))| \, d_q s \right. \\ & \left. t\left[|\theta| + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,x(s))| \, d_q s \right] \right). \end{aligned}$$

Using (9), we can write
$$\|O(x)\| \leq N\theta + \frac{2Nkr + 2NM}{\Gamma_q(\alpha+1)}.$$

Consequently,
$$\|O(x)\| \leq r,$$

which implies that $TB_r \subset B_r$.

Next, for $x, y \in X$ and, for each $t \in J$, we have :

$$\begin{aligned} & \|O(x) - O(y)\| \\ \leq\ & N \times \sup_{t \in J} \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s)) - f(s, y(s))| \, d_q s \right. \\ & \left. + t \left[\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s)) - f(s, y(s))| \, d_q s \right] \right). \end{aligned}$$

Thanks to (H_1), we can write

$$\begin{aligned} & \|O(x) - O(y)\| \\ \leq\ & N \times \sup_{t \in J} \left\{ \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} k \|x-y\| \, d_q s \right. \right. \\ & \left. + t \left[\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} k \|x-y\| \, d_q s \right] \right) \end{aligned}$$

This implies that

$$\|O(x) - O(y)\| \leq \frac{2Nk}{\Gamma_q(\alpha+1)} \|x-y\|.$$

In view of condition $\frac{2Nk}{\Gamma_q(\alpha+1)} < 1$, it follows that O is a contraction. So Banach's fixed point theorem applies and hence the operator O has a unique fixed point. This, implies that the problem (1) has a unique solution on J. This completes the proof.

4. Example

Example 10 *Consider the following $q-$fractional hybrid boundary value problem*

$$\begin{cases} D_q^{\frac{3}{2}} \left(\frac{x(t)}{\frac{1}{2}|\cos x(t)|+1} \right) = \frac{e^{-t}}{25+t^2} + \frac{1}{35} e^{-5t} \sin x(t), \ t \in [0,1], \ q = \frac{1}{2}, \\ \\ x(0) = 0, \ x(1) - 3 = 0. \end{cases} \tag{10}$$

For this example, we have $\alpha = \frac{5}{3}, q = \frac{1}{2}, \theta = \frac{5}{7}$ and for $x \in \mathbb{R}, \ t \in [0,1]$, we have

$$f(t, x(t)) = \frac{e^{-t}}{25+t^2} + \frac{1}{35} e^{-5t} \sin x(t),$$

and

$$g(t, x(t)) = \frac{1}{2} |\cos x(t)| + 1.$$

Moreover for any $x \in \mathbb{R}$, and any $t \in [0,1]$, we have

$$\begin{aligned} |f(t, x(t))| &\leq \frac{1}{25} + \frac{1}{35} |x|, \\ |g(t, x(t))| &\leq \frac{3}{2}. \end{aligned}$$

We can take, $\rho = \frac{1}{25}$, $\sigma = \frac{1}{35}$, $N = \frac{3}{2}$ *and* $2N\sigma = 0.085 < \Gamma_q(\alpha+1) = 1.2593.$

It is clear that the conditions of Theorem 8 *hold. Then the problem (10) has at least one solution on* $[0,1]$.

Example 11 *Let us consider the following* $q-$*fractional hybrid boundary value problem*

$$\begin{cases} D_q^{\frac{3}{2}}\left(\frac{x(t)}{\frac{1}{3}|\sin x(t)|+2}\right) = \frac{5}{31}\tan^{-1} x(t) + \cos(e^t), \ t \in [0,1], \ q = \frac{1}{2}, \\ \\ x(0) = 0, \ x(1) - \frac{2}{5} = 0. \end{cases} \tag{11}$$

Here $\alpha = \frac{3}{2}$, $q = \frac{1}{2}$, $\theta = \frac{2}{5}$ *and for* $x \in \mathbb{R}$, $t \in [0,1]$, *we have*

$$\begin{aligned} f(t, x(t)) &= \frac{5}{31}\tan^{-1} x(t) + \cos\left(e^t\right), \\ g(t, x(t)) &= \frac{1}{3}|\sin x(t)| + 2. \end{aligned}$$

Note that

$$|f(t,x) - f(t,y)| \leq \frac{5}{31}|x-y|, \ \ |g(t,x(t))| \leq \frac{7}{3},$$

and

$$\frac{2Nk}{\Gamma_q(\alpha+1)} = 0.632\,5 \ < 1.$$

Thus all the conditions of Theorem 9 are satisfied and, consequently, there exists a unique solution for the problem (11).

5. Conclusion

In this paper, a new intelligent method for fault detection in PV module is introduced. The case of partial shading effect is studied. The main advantage of the developed ANNBM is that doesn't require a complex system for the estimation of the photovoltaic module output power, neither a mathematical model, it can also detect any power decreasing carried out by a large types of failures that can be happened in the PV panel. This new strategy can be easily implemented in a numeric calculator using FPGA, and could also be integrated as a function for PV applications in a numeric instrument that will be our subject in the future works.

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