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Sigma-demicontractivity and Strong Convergence of Mann Iteration

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The problem of additional conditions ensuring the strong convergence of Mann iteration for demicontractive mapping is investigated. A critical analysis of σ-demicontractivity condition is performed showing the applicability and how strong this condition is.

***Keywords**: demicontractive mapping, Mann iteration, σ-demicontractivity*

1. Introduction.

Recall that the Mann iteration, or Krasnoselki-Mann iteration, has the form:

$$x_{n+1} = (1 - t_n)x_n + t_n T(x_n),$$

where T : C → C, C is a closed convex subset of the real Hilbert space *H* and $t_n \subset \mathbb{R}$ is the control sequence. The convergence properties of this iterative process have been widely studied and numerous results have been published, even in the last time [11]. It is worthwhile to note two interesting points in the development of Mann iterative studies, the both with negative assertions. The first [6], is an example of a contraction defined on a bounded closed convex subset of an Hilbert space for which the Krasnoselski iteration (the particular case of the Mann iteration with control sequence $t_n = 1/2$, n = 0, 1, . . .) does not converge. The second (Chidume and S. A. Mutangadura [5]), is an example of a Lipschitz pseudocontractive map for which the Mann iteration sequence fails to converge. The both examples show the difficulties and challenges in investigation the convergence properties of Mann iteration, this convergence depending on characteristics of spaces, properties of T and properties of control sequence [1].

To obtain strong convergence, some additional conditions are needed. The problem of finding such conditions was investigated by several authors, starting with the papers in which the concept of demicontractivity was introduced. Hicks and Kubicek [7] required as additional condition that $I - T$ maps closed bounded subsets of C into closed subsets of C; in particular, this is satisfied if T is demi-compact. The same type of compactness conditions (on the definition domain C or on the mapping are considered in more recently papers [3], [4].

In [9] the notion of σ-demicontractivity is introduced and it is shown that demicontractivity together with σ-demicontractivity ensure the strong convergence.

In this paper we collect some results concerning to the problem of σ- demicontractivity condition that ensures the strong convergence of the Mann iteration in real Hilbert spaces. We perform a critical analysis of this condition showing its strength and applicability.

2. Preliminaries

In the following $\mathscr{H}$ is a real Hilbert space (scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$), C is a closed convex subset of $\mathscr{H}$, T : C → C is a nonlinear mapping. We suppose everywhere that the set of fixed points of T is nonempty, Fix(T) ≠ ∅. Recall that the mapping T is said to be demicontractive with contraction coefficient k if:

$$\|T(x) - p\|^2 \le \|x - p\|^2 + k\|T(x) - x\|^2, \forall x \in C, p \in Fix(T) \tag{1}$$

Note that (2) is equivalent with:

$$\langle x - T(x), x - p\rangle \ge \lambda\|x - T(x)\|^2, \forall x \in C, p \in Fix(T) \tag{2}$$

where λ = (1 − k)/2.

The following lemma will be used in the sequel.

Lemma 2.1: The set of fixed points Fix(T) of a demicontractive mapping with the contraction coefficient $0 < k < 1$, is closed and convex.

Proof: To prove that Fix(T) is closed, let $\{x_n\}$ be an arbitrary sequence in Fix(T) which converges to a point p. We prove that p ∈ Fix(T). Observe that $T(x_n)=x_n$, ∀ n ≥ 0, and $\|x_n - p\| \to 0$ as n → ∞. We also have:

$$\|p - T(p)\| \le \|p - x_n\| + \|x_n - T(p)\|$$

Furthermore,

$$\|x_n - T(p)\|^2 = \|T(x_n) - T(p)\|^2 \le \|x_n - p\|^2 + k\|p - T(p)\|^2$$

and hence

$$\|x_n - T(p)\| \le \|x_n - p\| + \sqrt{k}\|p - T(p)\| \le \left(1 + \sqrt{k}\right)\|x_n - p\| + \sqrt{k}\|x_n - T(p)\|.$$

It follows that

$$\|x_n - T(p)\| \leq \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x_n - p\| \to 0 \ as \ n \to \infty \qquad (3)$$

Inequality (3) now implies that $p \in$ Fix(T). To prove that Fix(T) is convex let $p_1, p_2 \in$ Fix(T) be arbitrary, and let $p = (1 - \lambda)p_1 + \lambda p_2$ for arbitrary $\lambda \in [0, 1]$. We prove that $p \in$ Fix(T). Observe that $p - p_1 = \lambda (p_2 - p_1)$ and $p - p_2 = (1-\lambda)(p_1 - p_2)$ and:

$$\begin{aligned}\|p - T(p)\|^2 &= \left\|(1-\lambda)\big(p_1 - T(p)\big) + \lambda(p_2 - T(p)\right\|^2 \\ &= (1-\lambda)\|p_1 - T(p)\|^2 + \lambda\|p_2 - T(p)\|^2 - \lambda(1-\lambda)\|p_2 - p_1\|^2 \\ &\leq (1-\lambda)\|p - p_1\|^2 + k\|p - T(p)\|^2 - \lambda(1-\lambda)\|p_2 - p_1\|^2 \\ &= (1-\lambda)[\lambda^2\|p_2 - p_1\|^2 + k\|p - T(p)\|^2] \\ &+ \lambda[(1-\lambda)^2\|p_2 - p_1\|^2 + k\|p - T(p)\|^2] - \lambda(1-\lambda)\|p_2 - p_1\|^2\end{aligned}$$

hence $(1-k)\|p - T(p)\|^2 \leq 0$ and thus $p \in$ Fix(T).

3. σ-demicontractivity

In [9] is introduced the concept of *σ-demicontractivity*, in order to define a new additional condition for strong convergence of Mann iteration. A mapping T: C → C is said to be σ-demicontractive if Fix(T) ≠ ∅ and there exists $\lambda > 0$ and $\sigma > 1$ such that $\forall x \in C$, $\forall p \in$ Fix(T):

$$\langle x - T(x), x - \sigma p\rangle \geq \|x - T(x)\|^2 \qquad (4)$$

It can easily be observed that $\forall x \in C$, $\forall p \in$ Fix(T), (4) is equivalent to:

$$\|T(x) - \sigma p\|^2 \leq \|x - \sigma p\|^2 + k\|x - T(x)\|^2 \qquad (5)$$

where $k = 1 - 2\lambda$.

If $p \in$ Fix(T) then from (8) it follows $\sigma p \in$ Fix(t), therefore all points of the form $\sigma^j p$, j = 1, 2, . . . belong to Fix(T). If Fix(T) is a convex set, then the line segment $[p, \sigma^j p) \subset$ Fix(T). It is proved the following theorem [9]:

Theorem 3.1 Let T: C → C be a nonlinear mapping, where C is a closed convex subset of a real Hilbert space H. Suppose the following conditions are satisfied:

(i) I − T is demiclosed at 0.

(ii) T is demicontractive with contractive coefficient k, or equivalently T satisfies condition (7) with $\lambda = 1-k$

(iii) $0 < a \leq t_n \leq b < 2\lambda = 1 - k$.

Suppose in addition that T is σ-demicontractive for some $\sigma > 1$. Then for suitable x_0 the sequence $\{x_n\}$ given by Mann iteration converges strongly to an element of Fix(T).

The main steps of the proof are: From (i)-(iii) it follows that the sequence

$\{x_n\}$ converges weakly to a point $p \in Fix(T)$. Then let $x_0 \in C$ be such that $\langle x_0 - p, p\rangle \geq 1/[2(\sigma-1)]\|x_0 - p\|^2$, assuming that $\langle x_n - p, p\rangle \geq 1/[2(\sigma-1)]\|x_n - p\|^2$ and using the additional condition (T is σ-demicontractive) it can be proved that $\langle x_{n+1} - p, p\rangle \geq 1/[2(\sigma-1)]\|x_{n+1} - p\|^2$. Thus the induction is complete, and taking into account that $\{x_n\}$ converges weakly towards p, it results also the strong convergence.

Remark 3.2: The implicit meaning of the sentence "for suitable x_0" is: it can be chosen an x_0 such that the sequence $\{x_n\}$ starting with this x_0 converges weakly towards p, and that this two elements x_0, $p \in C$ satisfies the base of the induction $\langle x_0 - p, p\rangle \geq 1/[2(\sigma-1)]\|x_0 - p\|^2$. The existence of x_0 is not assured, and it should be either proved its existence in the particular cases or fond conditions in which x_0 exists.

Osilike observed ([10], Remark 2.3) that the condition of demicontractivity in *Theorem 3.1*, which is used only to prove the weak convergence of the sequence $\{x_n\}$, appears unnecessary since this convergence results from σ-demicontractivity. The proof follows the same general lines as the weak convergence for demicontractive case. We give below an improved version of *Theorem 3.1* and a brief outline of the proof.

Theorem 3.3: Suppose T: C → C is σ-demicontractive and that I − T is demiclosed at zero. Then Mann iteration with control sequence verifying $0 < a \leq t_n \leq b < 2\lambda$ converges strongly to an element of Fix(T).

Proof: Using (8) and the Mann iterative formula, it follows:

$$\|x_{n+1} - \sigma p\|^2 \leq \|x_n - \sigma p\|^2 - t_n(2\lambda - t_n)\|x_n - T(x_n)\|^2 \quad (6)$$

Therefore, from the restriction of $\{t_n\}$, we have:

$$\|x_{n+1} - \sigma p\| \leq \|x_n - \sigma p\|, n = 0,1,\ldots\, and\ \|x_n - \sigma p\| \to pp, n \to \infty$$

From (6) we obtain as well:

$$\|x_n - T(x_n)\|^2 \leq [a(2\lambda - b)]^{-1}(\|x_{n+1} - \sigma p\|^2 - \|x_n - \sigma p\|^2) \to 0\,.$$

The sequence being bounded, there exists a subsequence x_{nj} of $\{x_n\}$ which converges weakly towards an q. Since I − T is demiclosed at zero and

$|| x_{nj} - T(x_{nj})|| \to 0$, it results $q \in Fix(T)$. Suppose that there exits two subsequences, say $\{u_j\}$ and $\{v_j\}$ each converging weakly to u and v, respectively. As above, we have $||u_j - \sigma u|| \to \rho_u$ and $||v_j - \sigma v|| \to \rho_v$. Let us define e_j by:

$$e_j = ||u_j - \sigma u||^2 - ||v_j - \sigma v||^2 - ||u_j - \sigma u||^2 + ||v_j - \sigma v||^2$$

Obvious $e_j \to 0$, $j \to \infty$. On the other hand, we have

$e_j = -2\sigma\langle u_j - v - j, u - v\rangle \to 2\sigma\|u - v\|^2 and\ u = v$. All weakly convergent subsequence of $\{x_n\}$ converges to the same weak limit, therefore the sequence $\{x_n\}$ converges weakly to an element of Fix(T). The rest of the proof follows the lines the proof of Theorem 3.1.

We investigate in the sequel the shape of T and of the set of fixed points in the case of σ-demicontractive mapping. As before, we perform this in the particular case of a real function [8], f : [a, b] → ℝ. If p ∈ Fix(T) then $\sigma^j p$ ∈ Fix(T), j = 1, 2, . . . It can be proved, as in Lemma 2.1, that if p, q ∈ Fix(T) then the line segment [σp, σq]⊂ Fix(T), thus if p = minFix(T), then the line segment [σp, b] ⊂ Fix(T), as in Figure 1.

Note that the σ-demicontractive of a real function does not imply the convexity of its set of fixed points. For example, the function f : [0.3] → [0, 3], defined by:

$$f(x) = \begin{cases} 1, & x \le 1; \\ 2x - 1, & 1 < x < 1.25; \\ 1, & 1.25 < x < 1.50; \\ x, & x > 1.50 \end{cases} \quad (14)$$

is σ-demicontractive with σ = 1.5 and its set of fixed points is not convex.

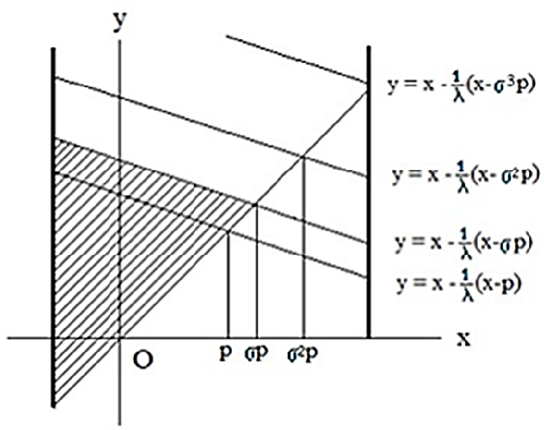

Figure 1. Case study of a real function

4. Conclusion

The σ-demicontractive condition is one of the most reasonable additional condition that ensure the strong convergence of the Mann iteration for demicontractive mappings. In the case of real function, sometimes, the σ-demicontractive condition does not imply the convexity of the set of fixed points.

References

[1] Artinescu I.M., *A note of the strong convergence of the Mann iteration for demicontractive mappings*, Applied Mathematical Sciences, 10, 255 – 261, 2016.

[2] Berinde V., *Iterative approximation of fixed points, Lecture Notes in Mathematics*, Springer, 2012.
[3] Boonchari D., Saejung S., *Construction of common fixed points of a countable family of -demicontractive mappings in arbitrary Banach spaces*, Appl. Math. Comput., 216, 173-178, 2010.
[4] Chidume C.E., Abbas M., Ali B., *Convergence of the Mann iteration algorithm for a class of pseudocontractive mappings*, Appl. Math. Comput. 194 (1), 1-6, 2007.
[5] Chidume C.E., Mutangadura S.A., *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc., 129, 2359-2363, 2001.
[6] Gene A.l., Lindenstraus J., *An example concerning fixed points*, Israel J. Math. 22 (1), 81-86, 1975.
[7] Hicks T.L., Kubicek J.D., *On the Mann iteration process in a Hilbert spaces*, J. Math. Anal. Appl. 59, 489-504, 1977.
[8] Hălălae I., Răduca E., Stroia M.D., *An analytical solution for the differential equation of the exciter of synchronous generators,* Annals of "Constantin Brancusi" University of Targu Jiu, Engineering series, No 2/2017, pp 82-87.
[9] Maruster L., Maruster St., S*trong convergence of the Mann iteration for α-demicontractive mappings*, Mathematical and Computer Modeling, 54 (9-10), 2486-2492, 2011.
[10] Osilike M.O., Strong and weak convergence of the Ishikawa iteration method for a class of nonlinear equations, Bull. Korean. Math., 37 (1), 153-169, 2000.
[11] Terfaloaga I.M., *The Convex Feasibility Problem and Mann-type Iteration*, Annals of "Eftimie Murgu" University of Reşiţa, Engineering series, 24 (1), 2017.

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