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Degree of Approximation of Function $\tilde{f} \in H_w$ Class by (E,1) (C,1) Means in the Holder Metric

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Abstract – In this paper, a theorem on degree of approximation of function $\tilde{f} \in H_w$ class by (E,1) (C,1) means in the Holder metric has been established.

Keywords - Degree of Approximation , Summability Method , Holder Metric, (E,1) mean, (C,1) mean.

1. Introduction

The degree of approximation of a function f belonging to various classes using different Summability method has been determined by many Mathematician ,Chandra [3] find the degree of approximation of function by Norlund transform .Later on Mahapatra and Chandra [4] obtain the degree of approximation in Holder metric using matrix transform .In sequal singh et.al. [7] obtain the error bound of periodic function in Holder metric again Mishra et.al. gave the generalization of result of Singh et.al. In this paper we find the degree of approximation of function $\tilde{f} \in H_w$ by (E,1) (C,1) means in holder metric.

2. Definition

For a 2π - periodic signal $f \in L^p$ periodic integrable in the sense of Lebesgue then the Fourier series of $f(x)$ is given by

$$f(x) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \qquad \text{…….(2.1)}$$

The conjugate series of Fourier series (2.1) is given by

$$\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx) \qquad \text{……(2.2)}$$

Let $w(t)$ and $w^*(t)$ denote two given modulai of continuity such that

$$(w(t))^{\frac{\beta}{\alpha}} = o\big(w^*(t)\big)\ as\ t \to 0^+\ for\ 0 \le \beta < \alpha < 1 \qquad \text{……..(2.3)}$$

Let $c_{2\pi}$ denote the Banach Space of all 2π - periodic continuous function defined on $[\pi, -\pi]$ under sub-norm the space L_p $[0,2\pi]$ $where\ p = \infty$ includes the space $c_{2\pi}$ For some positive constant k the function space H_w is defined by

$$H_w = \{f \in c_{2\pi} : |f(x) - f(y)| \le kw(|x-y|)\} \qquad \text{……(2.4)}$$

With norm $\|.\|_{w^*}$ defined by

$$\|f\|_{w^*} = \|f\|_c + \sup_{x,y}[\Delta^{w^*} f(x,y)] \qquad \text{……….(2.5)}$$

Where $w(t)$ and $w^*(t)$ are increasing function of t and

$$\|f\|_c = \sup_{0 \le x \le 2\pi}|f(x)| \quad \text{and} \quad \Delta^{w^*} f(x,y) = \frac{|f(x)-f(y)|}{w^*(|x-y|)} \quad x \ne y \qquad \text{…………(2.6)}$$

with the understanding that $\Delta^0 f(x,y) = 0$ If there exists positive constant β and k such that $w(|x-y|) \le \beta|x-y|^{\alpha}$ and $w^*(|x-y|) \le k|x-y|^{\beta}$ $0 \le \beta \le \alpha \le 1$ than the space

$$H_w = \{f \in c_{2\pi} : |f(x) - f(y)| \le k|x-y|^{\alpha} \ ,0 \le \alpha \le 1\} \qquad \text{.........(2.7)}$$

Is Banach space and metric induced by norm $\|.\|_{\alpha}$ and H_{α} is said to be Holder metric clearly H_{α} is a Banach space which decreases as α increases that is

$$H_{\alpha} \subseteq H_{\beta} \subseteq c_{2\pi} \ \text{ for } \ 0 \le \beta \le \alpha \le 1 \qquad \text{........(2.8)}$$

An infinite series$\sum_{n=0}^{\infty} a_n$ is said to be (C,1) summable to s if

$$(C,1) = \frac{1}{(n+1)} \sum_{n=0}^{\infty} s_k \ \to s \quad as \quad n \to \infty \qquad \text{........(2.9)}$$

The (E,1) transform is defined by

$$(E,1) = \frac{1}{2^n} \sum_{n=0}^{\infty} \binom{n}{k} s_k \to s \quad as \quad n \to \infty \qquad \text{.......(2.10)}$$

The (E,1) transform of (C,1) transform defined $(EC)_n^1$ is given by

$$(EC)_n^1 = \frac{1}{2^n} \sum_{n=0}^{\infty} \binom{n}{k} c_k^1 \to s \quad as \quad n \to \infty \qquad \text{......(2.11)}$$

3.Known Results

Singh and Mahajan [7] established the following theorem to error bound of signal passing through (C,1)(E,1) transform.

Theorem 1 – Let $w(t)$ defined (2.4) be such that

$$\int_t^{\pi} \frac{w(u)}{u^2} du = o\{H(t)\} \qquad H(t) \ge 0 \qquad \text{.......(3.1)}$$

$$\int_0^t H(u) du = o\{tH(t)\} \qquad as \ t \to 0^+ \qquad \text{.........(3.2)}$$

Then for $0 \le \beta < \alpha \le 1$ and $f \in H_w$ we have

$$\left\| t_n^{(CE)^1}(S; f) - f(x) \right\|_{w^*} = o\left\{ \left((n+1)^{-1} H(\tfrac{\pi}{n+1}) \right)^{1-\frac{\beta}{\alpha}} \right\} \qquad \text{........(3.3)}$$

Theorem 2 – Consider w(t) defined (2.4) and for $0 \le \beta \le \alpha \le 1$ and $f \in H_w$ we have

$$\left\| t_n^{(CE)^1}(f) - f(x) \right\|_{w^*} = o\left\{ \left(w(\frac{\pi}{n+1}) \right)^{1-\frac{\beta}{\alpha}} + ((n+1)^{-1} \sum_{k=1}^{n+1} w\left(\frac{1}{k+1}\right))^{1-\frac{\beta}{\alpha}} \right\}$$

.........(3.4)

In sequal Mishra and Khatri [11] gave the generalized result of above theorem . They proved the following .

Theorem 3 – Let $w(t)$ defined (2.4) be such that

$$\int_t^{\pi} \frac{w(u)}{u^2} du = o\{H(t)\} \qquad H(t) \ge 0$$

$$\int_0^t H(u)du = o\{tH(t)\} \qquad as\ t \to 0^+$$

Let Np be the Norlund summability matrix generated by the non –negative {Pn} such that (n+1)pn = o(Pn) $\forall n \geq 0.$

Then for $\bar{f} \in H_w$ $0 \leq \beta < \alpha \leq 1$ we have

$$\|t_n{}^{-NE}(f) - \bar{f}(x)\|w^* = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{\omega^*(|x-y|)}(\log(n+1))^{\frac{\beta}{\alpha}}\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\sigma}}\right\} \qquad \text{……..(3.5)}$$

And if w(t) satisfies (3.1) then for $\bar{f} \in H_w$ $0 \leq \beta < \alpha \leq 1$ we have

$$\|t_n{}^{-NE}(f) - \bar{f}(x)\|w^* = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{\omega^*(|x-y|)}\left(\log(n+1)\,w(\frac{\pi}{n+1})\right)^{1-\frac{\beta}{\alpha}} + \left(\left(\frac{1}{n+1}\right)\sum_{k=0}^{n} w\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\sigma}}\right\}$$

s …….(3.6)

4 .Main Theorem

In this paper we have to prove a theorem on the degree of approximation of a function f(x) conjugate to a 2π- periodic function f belonging to $\bar{f} \in H_w$ class by (E,1) (C,1) mean of conjugate series of its Fourier series.

Theorem 1 – Let $w(t)$ satisfy the following condition

$$\int_t^\pi \frac{w(u)}{u^2}du = o\{H(t)\} \qquad H(t) \geq 0 \qquad \text{…….(4.1)}$$

$$\int_0^t H(u)du = o\{tH(t)\} \qquad as\ t \to 0^+ \qquad \text{………(4.2)}$$

Then for $\bar{f} \in H_w$ $0 \leq \beta < \alpha \leq 1$ we have

$$\|t_n{}^{-EC}(f) - \bar{f}(x)\|w^* = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{\omega^*(|x-y|)}(\log(n+1))^{\frac{\beta}{\alpha}}\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\sigma}}\right\}$$

……….(4.3)

5 Lemma

In order to prove our main result ,we require the following lemma.

Lemma 1 - For $0 < t \leq \frac{\pi}{n+1}$ $\overline{K_n}(t) = o\left(\frac{1}{t}\right)$ ………(5.1)

Proof - - For $0 < t \leq \frac{\pi}{n+1}$, $sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $|\cos nt| \leq 1.$

$$\overline{K_n}(t) = \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(k+1)}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}$$

$$\leq \frac{1}{2^{n+1}t} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{k+1} \sum_{v=0}^{k} \right\}$$

$$= o\left(\frac{1}{t}\right)$$

Lemma 2 - For $\frac{\pi}{n+1} \leq t \leq \pi$ $\quad \overline{K_n}(t) = o\left(\frac{1}{t^2(n+1)}\right)$ ………(5.2)

Proof - For $\frac{\pi}{n+1} \leq t \leq \pi$, $sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $|\sin t| \leq 1$.

$$\overline{K_n}(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)} \sum_{v=0}^{k} \frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\}$$

$$\leq \frac{1}{2^{n+1}t} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)} \sum_{v=0}^{k} \cos\left(v+\frac{1}{2}\right)t \right\}$$

$$= \frac{1}{2^{n+1}t} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)} \left(\frac{-2\sin kt}{\sin\frac{t}{2}}\right) \right\}$$

.

$$\leq \frac{\pi}{2^{n+1}t^2} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1}$$

$$= o\left(\frac{1}{t^2(n+1)}\right)$$

Lemma 3 – If w(t) satisfies (4.1) and (4.2) then

$$\int_0^u t^{-1} w(t)dt = o\big(uH(u)\big) \qquad as \quad u \to 0^+ \qquad \text{……(5.3)}$$

Lemma 4 – If $\psi_x(t) = \psi(t) = f(x+t) - f(x-t)$ then for $\bar{f} \in H_w$ we get

$$\left|\psi_x(t) - \psi_y(t)\right| \leq 2M\, w(|x-y|) \qquad \text{…….(5.4)}$$

$$\left|\psi_x(t) - \psi_y(t)\right| \leq 2M\, w\,(|t|) \qquad \text{…….(5.5)}$$

6. Proof of Theorem

Let $\bar{s_n}(f;x)$ denote the partial sum of series $\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx)$. Then we have

$$\bar{s_n}(x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi_x(t) \frac{cos\left(n+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)} dt \qquad \text{….(6.1)}$$

The (C,1) mean of $\bar{s_n}(f;x)$ is given by

$$\overline{C_n^1} - \bar{f}(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi} \frac{\psi_x(t)}{sin\left(\frac{t}{2}\right)} \sum_{k=0}^{n} cos\left(k+\frac{1}{2}\right)t\, dt \qquad \text{…….(6.2)}$$

Now (E,1) (C,1) transform of $\bar{s_n}(f;x)$ is denoted by t_n^{-EC}we can write as

$$t_n^{-EC}(f) - \bar{f}(x) = \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^{\pi} \frac{\psi_x(t)}{sin\left(\frac{t}{2}\right)}\left(\frac{1}{k+1}\right)\left\{\sum_{v=0}^{k} cos\left(v+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \int_0^{\pi} \psi_x \overline{K}_n(t)dt \qquad \text{…..(6.3)}$$

Where $\bar{K}_n(t) = \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\left\{\binom{n}{k}\left(\frac{1}{k+1}\right)\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}$ …(6.4)

$$E_n(x,y) = |E_n(x) - E_n(y)| = \int_0^{\pi}\left|\psi_x(t) - \psi_y(t)\right|\bar{K}_n(t)dt$$

$$=\left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}.\right]\left|\psi_x(t) - \psi_y(t)\right|\bar{K}_n(t)dt$$

$$=I_1 + I_2 \quad \text{(Say)} \qquad \text{….(6.5)}$$

Using (5.5) and (5.1) assume that w(t) satisfies (4.1) and (4.2); we get

$$I_1 = \int_0^{\frac{\pi}{n+1}}\left|\psi_x(t) - \psi_y(t)\right|\bar{K}_n(t)dt$$

$$= \int_0^{\frac{\pi}{n+1}} t^{-1}\, w(t)\, dt$$

$$= o((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right) \qquad \text{….(6.6)}$$

Using (5.5) and (5.2) assume that w(t) satisfies (4.1) and (4.2); we get

$$I_2 = \int_{\frac{\pi}{n+1}}^{\pi}\left|\psi_x(t) - \psi_y(t)\right|\bar{K}_n(t)dt$$

$$= o\left(\frac{1}{n+1}\right)\int_{\frac{\pi}{n+1}}^{\pi} t^{-2}\, w(t)\, dt$$

$$= o((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right) \qquad \text{….(6.7)}$$

Now using (5.4) and (5.1) we get

$$I_1 = \int_0^{\frac{\pi}{n+1}}\left|\psi_x(t) - \psi_y(t)\right|\bar{K}_n(t)dt$$

$$= o(w|x-y|)\int_0^{\frac{\pi}{n+1}} t^{-1}\, dt$$

$$= o(w|x-y|\log(n+1)).$$

Again using (5.4) and (5.2) we get

$$I_2 = \int_{\frac{\pi}{n+1}}^{\pi} \left|\psi_x(t) - \psi_y(t)\right| \bar{K}_n(t)dt$$

$$= o\left(\frac{w|x-y|}{n+1}\right) \int_{\frac{\pi}{n+1}}^{\pi} t^{-2}\, dt$$

$$= o(w|x-y|\,). \quad \text{……..(6.9)}$$

Using the fact that we can write $I_k = I_k^{1-\frac{\beta}{\alpha}} I_k^{\frac{\beta}{\alpha}}$, $k = 1,2$

Combining(6.6) and(6.8) we get

$$I_1 = o\left(\left[((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right)\right]^{1-\frac{\beta}{\alpha}} [(w|x-y| \log(n+1))]^{\frac{\beta}{\alpha}}\right) \quad \text{……(6.10)}$$

Combining(6.7) and(6.9) we get

$$I_2 = o\left(\left[((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right)\right]^{1-\frac{\beta}{\alpha}} [(w|x-y|\,)]^{\frac{\beta}{\alpha}}\right) \quad \text{……(6.11)}$$

Now from (2.7),(6.10) and (6.11) we have

$$\sup_{x,y}\left|\Delta^{w^*} E(x,y)\right| = \sup_{x,y} \frac{\left|E_{n(x)} - E_{n(y)}\right|}{w^*(|x-y|)}$$

$$= o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{w^*(|x-y|)} (\log(n+1))^{\frac{\beta}{\alpha}} \left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\alpha}}\right\} \quad \text{….(6.12)}$$

Since $\|E_n(x)\|_c = \sup_{0 \le x \le 2\pi} \left|t_n^{-EC}(f) - \bar{f}(x)\right|$ ….(6.13)

From (6.6) and (6.7) we get

$$\|E_n(x)\|_c = o\left(((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right)\right) \quad \text{…..(6.14)}$$

Combining (6.12) and (6.14) we get

$$\left\|t_n^{-EC}(f) - \bar{f}(x)\right\|_{w^*} = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{w^*(|x-y|)} (\log(n+1))^{\frac{\beta}{\alpha}} \left(((n+1)^{-1})\, H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\alpha}}\right\}$$

This complete the proof of theorem.

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