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NUMERICAL METHODS FOR SOLVING ONE VARIABLE AND SYSTEMS OF NONLINEAR EQUATIONS

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ABSTRACT

In this paper we will focus on the numerical methods involved in solving nonlinear equation in one variable and systems of nonlinear equations. First we will study the fixed point iteration and Newton's method in one variable for solving nonlinear equation and their convergence. Second we will examine these two methods in for solving of multivariable nonlinear equations which involves the Jacobean matrix and finally we also give an application of Newton's methods.

KEYWORDS: Numerical methods of solving nonlinear equations.

INTRODUCTION

In general it is not possible to determine a zero of ξ of a function $f : E \to E$ explicit with a finite number of steps, so we have to restore to approximation methods. This method are usually iterative [i.e. a trial of sequence] and have the following form.

Beginning with starting value of $X^{(0)}$ successive approximation $X^i, i = 1,2,\cdots to\, \xi$ are computed with aid of an iteration function $G : E \to E$ such that $X^{i+1} = G(X^i)$.

The Following Question Arises in this Connection According to [4].

How is a suitable iteration function to be found?

Under what conditions the sequence X^i will converges?

How quickly will the sequence X^i converges

In this seminar is to examine two different numerical methods that are used to solve nonlinear equation in one variable and systems of nonlinear equations in several variables. The first method will look at the fixed point iteration methods this will be followed by Newton's methods. For each method, a breakdown of each numerical procedure and theorems will be proved. In addition, there will be some discussion of the convergence of the numerical methods, as well as the advantages and disadvantages of the Newton's methods.

PRELIMINARIES

Some useful Definition and Theorems

Definition-1 The general form of a system of non linear system of equation is:

$$\begin{cases} f_1(x_1,\cdots,x_n)=0 \\ f_2(x_1,\cdots,x_n)=0 \\ \quad\vdots \\ f_n(x_1,\cdots,x_n)=0 \end{cases} \quad \textbf{(1.1)}$$

Where each $f_i, i=1,2,...,n$ be mapping a vector $x=(x_1,\cdots,x_n)^T$ of the n-dimensional space R^n into the real line.

The system can be alternatively be represented by defining a function mapping R^n into R^n by:

$$F(x_1,\cdots,x_n)=(f_1(x_1,\cdots,x_n),\cdots,f_2(x_1,\cdots,x_n))^T$$

Using vector notation to represent the variables $x_1,x_2,\cdots,x_n$ system (1.1) assumes the form

F(x)=0 **(1.2)**

The function $f_1,f_2,\cdots,f_n$ are called the co ordinate function of F.

Example: The three-by-three system of nonlinear equation

$$\begin{cases} 3x_1-\cos(x_2)x_3-\dfrac{1}{2}=0 \\ x^2{}_1-81[x_1+0.1]^2+\sin(x_3)+1.06=0 \\ e^{-x_1x_2}+20x_3+\dfrac{10\pi-3}{3}=0 \end{cases}$$

The above equation can be replaced in the form by defining the three function $f_1,f_2, and\ f_3$ from $R^3\ into\ R$

$$\begin{cases} f_1(x_1,\cdots,x_n)=3x_1-\cos(x_2)x_3-\dfrac{1}{2} \\ f_2(x_1,\cdots,x_n)=x^2{}_1-81[x_1+0.1]^2+\sin(x_3)+1.06 \\ f_3(x_1,\cdots,x_n)=e^{-x_1x_2}+20x_3+\dfrac{10\pi-3}{3} \end{cases}$$

And $f\ R^3 \to\ R^3$

$$F(x_1,\cdots,x_n)=(f_1(x_1,\cdots,x_n),f_2(x_1,\cdots,x_n),f_3(x_1,\cdots,x_n))^T$$
$$=(3x_1-\cos(x_2)x_3-\frac{1}{2},x^2{}_1-81[x_1+0.1]^2+\sin(x_3),e^{x_1x_2}+20x_3+\frac{10\pi-3}{3})^T$$

Defination-2 : Let f be a function define on a set D$\subseteq R^n$ and mapping into R.The function f is said to have the limit L at x_0 written $\lim_{x\to x_0} f(x) = L$ if given any number $\varepsilon > 0$, a number $\delta > 0$ exits with property that $|f(x) - L| < \varepsilon$ whenever $x_0 \in D$ and $\|x - x_0\| < \delta$

Mean Value Theorem

If $f \in [a,b]$ and f is differentiable ob [a, b], then a number ξ in such that $a < \xi < b$ exists such that $f'(\xi) = \frac{f(b) - f(a)}{b-a}$ where C[a,b] is the set of all continuous function on [a,b]

Intermediate Value Theorem

If $f \in [a,b]$ and y is any number between f (a) and f(b),then there is x in (a,b) for which f(x)=y

Jacobean Matrix

Let E be n-dimensional with a base $e_1, e_2, \cdots, e_n$ and corresponding coordinate $x_1, x_2, \cdots, x_n$ and Let F be m-dimensional with base $e_1^-, e_2^-, \cdots, e^-_m$ and corresponding coordinate $x^-{}_1, \cdots, x^-{}_m$.Let G$\subset E$ be open set and let $f : G \to F$,in terms of coordinate, we may written the mapping f as

$$\begin{aligned} x_1^- &= f_1(x_1, \cdots, x_n) \\ x_2^- &= f_2(x_1, \cdots, x_n) \\ &\vdots \\ x_n^- &= f_n(x_1, \cdots, x_n) \end{aligned}$$

If f s differentiable at $X \in G$,then it is differentiable at X is given by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Hessian Matrix

The Hessian matrix, will be discussed in a future proof

Definition: The Hessian matrix is a matrix of second order partial derivative $H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j}$

Such that

$$H(x)=\begin{bmatrix} \frac{\partial^2 f}{\partial x^2{}_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n{}^2} \end{bmatrix}$$

Norm

Definition 3: A norm on R^n is a real –valued function $\|.\|$ define on R^n and satisfying the three conditions below [where before O denote the zero vector in R^n]

1. $\|x\| \geq 0 \ and \ \|x\| = 0 \ iff \ \ x = 0$

2. $\|\alpha x\| = |\alpha|\|x\|$,for any scalar α and vector x.

3. $\|x+y\| \leq \|x\| + \|y\|$,for all vectors x and y

The quantity $\|x\|$ is thought of as being a measure of the size of the vector x and the double bar are used to emphasize the distinction between the size of the vector and the absolute value of a scalar.

Three useful examples of norms are the so-called l_p norms $\|.\|_p \ for \ R^n, p = 1,2,\infty$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|\right)^{\frac{1}{2}}$$

$$\|x\|_\infty = \max\{|x_i|\}, i = 1,2,\cdots,n$$

Norm of Matrix

Definition 4: Let M_n denote the set of all (nxn) matrix and let O denoted the (nxn) zero matrix. Then a matrix norm for M_n is real valued function $\| \ \|$ which define on M_n and will satisfy the following condition for all (nxn) matrices A and B

1. $\|A\| \geq 0 \ and \ \|A\| = 0 \ \ iff \ \ A = 0$

2. $\|\alpha A\| = |\alpha|\|A\|$,for any scalar α

3. $\|A+B\| \leq \|A\| + \|B\| \quad and \quad \|AB\| \leq \|A\|\|B\|$

Just as there are numerous way of defining specific vector norm, there also way defining specific matrix norm. We concentrate three basic matrix norm specially if $(a_{ij}) \in M_n$ .Then

1. $\|A\|_1 = \underset{1\le j\le n}{Max}\left[\sum_{i=1}^{n}|a_{ij}|\right] \rightarrow$ Maximum absolute column sum

2. $\|A\|_\infty = \underset{1\le i\le n}{Max}\left[\sum_{j=1}^{n}|a_{ij}|\right] \rightarrow$ Maximum absolute row sum

3. $\|A\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}$

General Convergence

A sequence of vectors $\{x_n\} \in R^n$ converges to a vector x ,if for each $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that $\|x_n - x\| < \varepsilon \ \ for\ all\ n \ge N(\varepsilon)$.

In order to characterize the speed of convergence of a convergent sequence x_n $\lim_{n\to\infty} x_n = x$,we say that the sequence converges at least with order $p \ge 1$ if there is a constant c>0 with if c<1 if $p \ge 1$ and an integer N such that the inequality $\|x_n - x\| \le C\|x_n - x\|^p$,$0 < C < 1$ holds for all $i \ge N$

i) I f $p = 1$, then order of convergence is linear.

ii) If $p = 2$, then the order of convergence is quadratic.

The speed of convergence increase with decrease C.Therefore C is called convergence factor

Taylors Theorem for function of one variable

If f(x) is continuous and posses continuous derivative of order n in an interval includes x=a,then in that interval

$f(x) = f(a) + (x-a)f'(a) + (x-a)^2\dfrac{f''(a)}{2!} + \cdots + \dfrac{(x-a)^{n-1}}{(n-1)!}f^{(n)}(a) + R_n(x)$,the remainder term can be expressed in the form $R_n(x) = \dfrac{(x-a)^n}{n!}f^{(n)}(\xi), a < \xi < x.$

Taylor's series for function of several variables

$$f(x_1+\Delta x_1, x_2+\Delta x_2, \cdots, x_n+\Delta x_n) = f(x_1, x_2, \cdots, x_n) + \frac{\partial f}{\partial x_1}\Delta x_1 + \cdots +$$

$$\frac{\partial f}{\partial x_n}\Delta x_n + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1{}^2}(\Delta x_1)^2 + \cdots + \frac{\partial^2 f}{\partial x^2{}_n} + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \cdots + 2\frac{\partial^2 f}{\partial x_{n-1}\partial x_n}\Delta x_{n-1}\Delta x_n\right] + \cdots$$

ITERATIVE METHODS FOR NONLINEAR EQUATION IN ONE VARIABLE

Fixed –point iteration

We consider methods for determining the solution to an equation that is expressed for some iteration function in the form:

$$g(x) = x \quad whenever \quad f(x) = 0 \text{ -- (2.1)}$$

A solution to such an equation (2.1) is said to be a fixed – point of the function g.If a fixed point could be found for any given g, then the every root finding problem could be solved.

The construction of the iteration function g is not unique

For example if $f(x) = x^3 - 13x + 18 = 0$

Then possible choice for g(x) might be to list a few:

- $g(x) = \dfrac{(x^3 + 18)}{13}$
- $g(x) = (13x - 18)^{\frac{1}{3}}$
- $g(x) = \dfrac{(13x^2 - 18)}{x^2}$

In this case if $f(\xi) = 0 \ iff \ \xi = g(\xi) \ and \ \xi$ **is said to be a fixed point of g(x).**

Starting with a suitable initial value x_0 to expected root, compute

$$x_1 = g(x_0),\ x_2 = g(x_1),\ \cdots \ in\, general \quad x_{n+1} = g(x_n)$$

The sequence $\{x_n\}$ of numbers converges to ξ under a certain condition and this ξ is the required solution.This method is called **fixed point** method.

Definition 2.1: A point ξ is called a fixed point of iteration function g(x) iff $g(\xi) = \xi$.

The following theorem gives sufficient condition for existence and uniqueness of fixed point.

The following Theorem taken from [6]

Theorem 2.1: [Fixed point theorem]

Let g(x) be an iteration function define on the interval I =[a,b] such that

- $g(x) \in I \text{ for all } x \in I$
- $a \le g(x) \le b \text{ for all } x, a \le x \le b$
- The function g(x) is differentiable on I (implies continuity on I) and there exists a positive number k<1 such that $|g'(x)| \le k < 1 \text{ for all } x \in I$

Then

- g(x) has a fixed point ξ in I
- fixed point ξ is unique.
- Sequence generated by the rule $x_{n+1} = g(x_n)$ starting with initial $x_0 \in I$ converges to the fixed point ξ.

PROOF

Existence

If $g(a) = a \text{ or } g(b) = b$,the existence of a fixed point is obvious.

Suppose not. Then it must true that $g(a) > a \text{ and } g(b) < b$. Define $h(x) = g(x) - x$,h is continuous on [a,b] since g(x) and x are continuous on [a,b].More over h(a)=g(a)-a>0 and h(b)=g(b)-b<0.By intermediate value theorem there exists ξ in (a,b) for which

$$h(\xi) = g(\xi) - \xi = 0 \Rightarrow g(\xi) = \xi$$

Therefore g has fixed point.

Uniqueness

Suppose $\xi_1 \text{ and } \xi_2$ are two distinct points of g(x) in I, then $\xi_1 = g(\xi_1) \text{ and } \xi_2 = g(\xi_2)$.

Then $\xi_1 - \xi_2 = g(\xi_1) - g(\xi_2)$ and by mean value theorem for some $\alpha \in (a,b)$.

$$|\xi_1 - \xi_2| = |g'(\alpha)(\xi_1 - \xi_2| \le |g'(\alpha)||\xi_1 - \xi_2| \le k|\xi_1 - \xi_2|$$
$$\Rightarrow |\xi_1 - \xi_2| < |\xi_1 - \xi_2| \text{ since } |g'(\alpha)| \le k < 1$$

A positive number less than itself which is impossible.

Therefore, the assumption that $\xi_1 \text{ and } \xi_2$ are distinct is false

Hence $\xi_1 = \xi_2$

I.e. fixed point is unique.

Convergence

From (a) a fixed point ξ of g(x) exits $g(\xi)=\xi$

$x_{n+1}=g(x_n)$ Since g(x0 iteration function

$\xi = g(\xi)$, ξ is a fixed point of g(x)

$\Rightarrow \xi - x_{n+1} = g(\xi) - g(x_n)$

$\Rightarrow e_{n+1} = (\xi - x_n)g'(\alpha_n)$, α_n lies between x_n *and* ξ by mean value theorem

$e_{n+1} = e_n g'(\alpha_n) \le ke_n \Rightarrow |e_{n+1}| \le k|e_n|$

$\Rightarrow |e_{n+1}| \le k|e_n|$
$\le k^2|e_{n-1}|$ By using the above in quality

$\le k^3|e_{n-2}|$
$\vdots$

$\Rightarrow |e_{n+1}| \le k^{n+1}|e_0|$

$$\lim_{n\to\infty}|e_{n+1}| = 0 \text{ since } \lim_{n\to\infty} k^{n+1} = 0$$

$$\lim_{n\to\infty}|\xi - x_{n+1}| = 0$$

$$\Rightarrow \lim_{n\to\infty} x_{n+1} = \xi$$

Therefore the sequence converge to ξ

Theorem 2.2: Let g(I) $\subseteq I$ *and* $|g'(x)| \le k < 1$ *for all* $x \in I$. For $x_0 \in I$, the sequence $X_{n+1} = g(x_n)$, n=1,2,…converges to a fixed point ξ and the n^{th} *error* $e_n = x_n - \xi$ and satisfies

$$e_n = \frac{k^n}{1-k}|x_1 - x_0|$$

Proof we proved the convergence a sequence $\{x_n\}$ converges to a fixed point ξ.Now we want to prove the n^{th} error estimation.

From the sequence $x_{n+1} = g(x_n)$

Now $|x_2 - x_1| = |g(x_1) - g(x_0)| \le k|x_1 - x_0|$ [using mean value theorem]

$\Rightarrow |x_2 - x_1| \le k|x_1 - x_0|$

And $|x_3 - x_2| \le |g(x_2) - g(x_1)| \le k|x_2 - x_1| \le k^2|x_1 - x_0|$

$\Rightarrow |x_3 - x_2| \le k^2|x_1 - x_0|$

$\vdots$

$|x_{n+1} - x_n| \le k^n|x_1 - x_0|$, [By induction]

By several application of triangle inequality we get,

$$|x_{n+m} - x_n| \le \sum_{r=1}^{m}|x_{n+r} - x_{n+r-1}|$$

$$\le \sum_{r=1}^{m} k^{n+r-1}|x_1 - x_0|$$

$$\le k^n|x_1 - x_0|\sum_{r=1}^{m} k^{r-1}$$

$\Rightarrow \left|\lim_{m\to\infty} x_{n+m} - x_n\right| \le k^n|x_1 - x_0|\lim_{m\to\infty}(\sum_{r=1}^{m} k^{r-1}$),since k<1

$\Rightarrow |\xi - x_n| \le k^n|x_1 - x_0|\dfrac{1}{1-k}$,since $\lim_{m\to\infty} x_{n+m} = \xi$

Therefore $e_n \le \dfrac{k^n}{1-k}|x_1 - x_0|$

Convergence of Fixed Point Iteration

Theorem 2.3: if g(x) be the iteration function $x_{n+1} = g(x_n)$ is such that $g'(x)$ is continuous in some neighborhood of a fixed point ξ and $g'(\xi) \ne 0$,then fixed point method converges linearly.

Proof: Let e_n e the error in the n[th] approximation i.e $e_n = \xi - x_n$

$$e_{n+1} = \xi - x_{n+!}$$
$$= g(\xi) - g(x_n)$$
Then $$= g(\xi) - [g(\xi) - e_n g'(\xi) + higher\ order\ derivative\}$$
$$= e_n g'(\xi)$$
$$\left|\frac{e_{n+1}}{e_n}\right| = |g'(\xi)| = c$$

Therefore the fixed point convergence is linearly.

The following example taken from [4]

Example 1: Find the root of the function $f(x) = 2x - \cos x - 3 = 0$, correct to four decimal places on the interval $[-\pi, \pi]$

Solution: first rewrite the equation in the form x=g(x)

$$\Rightarrow x = \frac{\cos x + 3}{2}, the\, iteration\ \ function$$
$$g(x) = \frac{\cos x + 3}{2}$$

And using fixed point theorem $|g'(x)| = \left|\frac{\sin x}{2}\right| < 1, \forall x \in [-\pi, \pi]$.

Hence there exist a unique fixed point in $[-\pi, \pi]$ and we iterate using $x_{n+1} = g(x_n)$ with starting point $x_0 \in [-\pi, \pi]$.

Let $x_0 = \frac{\pi}{2}$

The Results are given in the Following Table 1

Table 1

N	x_n	$\lvert x_{n+1} - x_n \rvert$
0	$\frac{\pi}{2}$	-
1	1.5	$7x10^{-2}$
2	1.5354	$3.54x10^{-2}$
3	1.5177	$1.77x10^{-2}$
4	1.5265	$8.8x10^{-3}$
5	1.5201	$4.4x10^{-3}$
6	1.5243	$2.2x10^{-3}$
7	1.5232	$1.1x10^{-3}$
8	1.5238	$6x10^{-4}$
9	1.5235	$3x10^{-4}$
10	1.5236	$1x10^{-4}$

Therefore, the solution is $\xi = 1.5236$

Example 2: An engineer might want to find the pressure needed to cause a fluid suspension of particle to flow through a pipe, its diameter, the quantity of fluid that is to flow, and a number called the friction factor f that has been determined from experiment. The following nonlinear equation can be computed the friction factor f .

$$\frac{1}{\sqrt{f}} = \frac{1}{k}\ln(RE(\sqrt{f}) + 14 - \frac{5.6}{k} \quad (*)$$

Where the parameter k is known and RE is so called Reynolds number, can be computed from the pipe diameter, the velocity of the fluid. Then what is the value of f if k=0.28 and RE =3750.

Solution: first we have to find the suitable iteration function g from $(*)$ by analytic methods

$$g(f) = \frac{\dfrac{e^{0.14\sqrt{f}}}{f} + 0.84}{61.237}, let\ f_0 = 0.1\ and\ Tolerance(\varepsilon = 10^{-9})$$

The Results Given In the Following Data Table the Out Put the Results

Table 2

N	f_n	$\left\| f_{n+1} - f_n \right\|$
0	0.1	0.041106981
1	0.058893019	0.01788495
2	0.041008069	0.034508069
3	0.075516138	0.012560378
4	0.0629557595	$3.13x10^{-3}$
5	0.066087062	$8.78x10^{-4}$
6	0.065208764	$2.39x10^{-4}$
7	0.065447557	$6.55x10^{-5}$
8	0.065382071	$1.79x10^{-5}$
9	0.065399987	$4.9x10^{-6}$
10	0.065395082	$3.6x10^{-6}$
11	0.065396425	$1.34x10^{-6}$
12	0.065396057	$1.01x10^{-7}$
13	0.065396158	$2.7x10^{-8}$
14	0.065396131	$7x10^{-9}$
15	0.065396138	$2x10^{-9}$
16	0.065396136	$1x10^{-9}$

Therefore,f=0.065396136

Newton –Rapson Method of Iteration One Variable

Let x_0 be the approximation root of $f(x)$ *and* $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$ and expanding $f(x_0 + h)$ by Taylor's series we obtain:

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f(x_0 \ ) + higher\ derivative = 0$$

Neglecting the second and higher order derivative we have:

$$f(x_0) + hf'(x_0) = 0$$

Which gives? $h = -\dfrac{f(x_0)}{f'(x_0)}$

Hence x_1 a better approximation than x_0 is therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximation are given by $x_2, \cdots, x_n, x_{n+1}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Equation (2) which is called the Newtons Raphson formula

We indicate a geometric interpretation of Newtons method for the case that f(x), x_0 and the solution being sought are real. Geometrically, one step of Newton –Rapson method consists of replacing the curve y=f(x) the straight line $y = f(x_0) + f'(x_0)(x - x_0)$ which is tangent to the curve at the point $x = x_0$,the approximation x_{n+1} is the point of intersection of the tangent line the curve y=f(x) at the point $x = x_n$ with the X-axis.

Convergence of Newton-Rapson Method of Iteration

Theorem 2.4: if the iteration function g(x) is such that $g''(x)$ is continuous in some neighborhood of a fixed point ξ *and* $g'(\xi) = 0$, then Newton's method converges quadratic ally.

Proof: Now $e_{n+1} = \xi - x_{n+1} = g(\xi) - g(x_n)$

$$= g(\xi) - g(\xi - x_n) \ \text{since}\ x_n = \xi - e_n$$

$$= g(\xi) - \left[g(\xi) - e_n g'(\xi) + \frac{e^2{}_n g''(\xi)}{2} + higher\ order\ derivative \right]$$

Neglecting the higher order derivative, then we have

$$e_{n+1} = e^2{}_n \frac{g''(\xi)}{2}, \ \text{since}\ g(\xi) = 0 \ and \ g''(\xi) = k > 0$$

$$\Rightarrow \frac{|e_{n+1}|}{|e_n|^2} = \frac{k}{2}$$

Therefore the convergence of Newtons-Raphson iteration method is quadratic ally.

METHOD OF ITERATION FOR NONLINEAR SYSTEM OF EQUATION WITH N-EQUATION WITH N- VARIABLE

This section is concerned with the solution of n-simultaneous nonlinear equation in n –variable.

Such problems arise in a large varity of ways and varity of methods are necessarily to treat them.

We just introduce the subject give of these methods in chapter -2 fixed point iteration and Newton's methods in one variable.

Fixed point for n-equation with n-unknown

Suppose we have a system of nonlinear equation

$$\begin{cases} f_1(x_1,\cdots,x_n)=0 \\ f_2(x_1,\cdots,x_n)=0 \\ \vdots \\ f_n(x_1,\cdots,x_n)=0 \end{cases} \quad \textbf{(3.1)}$$

Suppose these equations have a solution $(\xi_1,\cdots,\xi_n)^T=\xi$

Re write this equation in equivalent form

$$\begin{cases} x_1=g_1(x_1,\cdots,x_n) \\ x_2=g_2(x_1,\cdots,x_n) \\ \vdots \\ x_n=g_n(x_1,\cdots,x_n) \end{cases} \quad \textbf{(3.2)}$$

The solution (3.1) and (3.2) has the same solution $(\xi_1,\cdots,\xi_n)^T=\xi$ and let $x^{(k)}=(x_1^{(k)},\cdots,x_n^{(k)})$ be the k[th] approximation of $(\xi_1,\cdots,\xi_n)^T=\xi$ with construction of the iteration scheme

$$\begin{cases} x_1^{(k+1)}=g_1(x_1^{(k)},\cdots,x_1^{(k)}) \\ \vdots \\ x_n^{(k+1)}=g_n(x_1^{(k)},\cdots,x_n^{(k)}) \end{cases} \quad , \ldots\ldots\ldots\ldots(3.3)$$

Equation (3.3) can be written in compact form

$$X^{(K+1)}=G(X^{(k)})-----------------------(3.4)$$

$$where\ X^{(K+1)}=(x_1^{(k+1)},\cdots,x_n^{(k+1)})^T,\ G(X^{(K)})=(g_1(x_1^{(k)}),\cdots,g_n(x^{(k)})^T\ and\ X^{(K)}=(x_1^{(k)},\cdots,x_n^{(k)})^T$$

For k=0, 1, 2…

The following theorem extends the fixed point theorem in chapter 2,to the n-dimensional case. This theorem is a special case of the well known contraction mapping.

Theorem: Let $D=\{(x_1,\cdots,x_n)^T/a_i\le x_i\le b_i,\ for\ each\ i=1,2,\cdots,n\}$ for some collection of constant $a_1,\cdots,a_n\ and\ b_1,\cdots,b_n$. Suppose G is a continuous function with partial derivative from $D\subseteq R^n$ with the property that $G(X)\in D\ whnever\ x\in D$. Then G has as fixed point in D,moreover,suppose a constant 0<m<1 exits with

$\left|\frac{\partial g_i(x)}{\partial x_j}\right| \le \frac{m}{n}, whenever \ \ x \in D$ for each j=1,2,…,n and each component function g_i. Then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ define by arbitrary selected $x^{(0)}$ in D and generated by $X^{(K+1)} = G(X^{(K)}) \ \ for\, each \ k \ge 0$ converges to the unique point $\xi \in D$ and $\left\|x^{(k)} - \xi\right\|_{\infty} \le \frac{m^k}{1-m}\left\|x^{(1)} - x^{(0)}\right\|_{\infty}$. Taken from [3]

Proof: The proof is similar to the fixed point theorem in chapter 2 the only difference we use the norm $\|.\|_{\infty}$

The following example taken from [3]

Example: Consider the following the nonlinear system of equations

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0 \\ x_1{}^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \end{cases}$$

If the i^{th} equation is solved for x_i,the system can be changed into the fixed-point problem

$$x_1 = \frac{1}{3}\cos(x_2 x_3) + \frac{1}{2}$$
$$x_2 = \frac{1}{9}\sqrt{x_1{}^2 + \sin x_3 + 1.06} - 0.1$$
$$x_3 = \frac{-1}{20}e^{-x_1 x_2} - (\frac{10\pi - 3}{60})$$

Let G: $R^3 \to R^3$ e define $G(x) = (g_1(x), g_2(x), g_3(x))^T$ and $X = (x_1, x_2, x_3)^T$,where

$$g_1(x_1, x_2, x_3) = \frac{1}{3}\cos(x_2 x_3) + \frac{1}{2}$$
$$g_2(x_1, x_2, x_3) = \frac{1}{9}\sqrt{x_1{}^2 + \sin x_3 + 1.06} - 0.1$$
$$g_3(x_1, x_2, x_3) = \frac{-1}{20}e^{-x_1 x_2} - (\frac{10\pi - 3}{60})$$

Using the above theorem (3.1) will be used to show that G a unique fixed point in

$$D = \{(x_1, x_2, x_3)^T / -1 \le x \le 1, \ for\, each\, i = 1,2,3\}$$

Now for $x = (x_1, x_2, x_3)^T \in D$

$$\left|g_1(x_1,x_2,x_3) \le \frac{1}{3}\left|\cos\right|(x_2x_3)\right| + \frac{1}{6} \le 0.5$$

$$\left|g_2(x_1,x_2,x_3)\right| \le \frac{1}{9}\sqrt{1+\sin(1)+1{,}06} + \left|0.1\right| \le 0.29 < 0.3$$

$$\left|g_3(x_1,x_2,x_3)\right| \le \left|\frac{1}{20}e^{-x_1x_2} + \frac{10\pi - 3}{60}\right| \le 0.61$$

So $-1 \le g_i(x_1,x_2,x_3) \le 1,\ for\ each\ i = 1,2,3.$ Thus $G(X) \in D\ whenever\ x \in D.$ Finding bounds for the partial derivative on D given the following.

$$\left|\frac{\partial g_1}{\partial x_1}\right| = \left|\frac{\partial g_2}{\partial x_2}\right| = \left|\frac{\partial g_3}{\partial x_3}\right| = 0$$

While, $\left|\frac{\partial g_1}{\partial x_2}\right| \le \frac{1}{3}\left|x_3\right|\left|\sin(x_2x_3)\right| \le 0.281$

$$\left|\frac{\partial g_2}{\partial x_3}\right| = \frac{\left|\cos x_3\right|}{18\sqrt{x_1{}^2 + \sin x_3 + 1.06}} \le 0.119$$

$$\left|\frac{\partial g_3}{\partial x_2}\right| = \left|\frac{x_2}{20}\right|e^{-x_1x_2} \le 0.14$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \left|\frac{x_1}{20}\right|e^{-x_1x_2} \le 0.14$$

Since the partial derivative of $g_1, g_2\ and\ g_3$ are bounded on D and $g_1, g_2\ and\ g_3$ are continuous on D consequently G is continuous on D.

Moreover, for every $x \in D, \left|\frac{\partial g_i(x)}{\partial x_j}\right| \le 0.281,\ for\ each\ j,i = 1,2,3$ and the condition in the second part of Theorem (3.1) holds m=0.281X3=0.843<1.

Therefore, G has a unique fixed point in D and the nonlinear system of equation has solution in D.

Note that G having a unique solution in D does not imply that the solution to the original system is unique on this domain. For example the solution x_2 in the above example involved the choice of the principal square root (the positive square root).

To approximate the fixed point ξ we will choose $x^{(0)} = (0.1, 0.1, -0.1)^T$ .The sequence of vectors generated by

$$x_1^{(k+1)} = \frac{1}{3}\cos(x_2^{(k)} x_3^{(k)}) + \frac{1}{2}$$

$$x_2^{(k+1)} = \frac{1}{9}\sqrt{(x_1^{(k)})^2 + \sin x_3^{(k)} \quad +1.06} - 0.1$$

$$x_3^{(k+1)} = \frac{-1}{20} e^{-x_1^{(K)} x_2^{(K)}} - (\frac{10\pi - 3}{60})$$

The sequence was generated until k was found with tolerance:

$$\varepsilon = \left\| x^{(k+1)} - x^{(K)} \right\|_\infty < 10^{-5}$$

The Results are Given in the Following Table

Table: 3

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\| X^{(K+1)} - X^{(K)} \right\|_\infty$
0	0.1000	0.1000	-0.1000	
1	0.49998333	0.00944115	0.52310127	0.423
2	0.499999593	0.00002557	0.52336331	$9.4x10^{-3}$
3	0.50000	0.00001234	0.52359847	$2.3x10^{-4}$
4	0.50000	0.000000003	-0.52359847	$1.2x10^{-5}$
5	0.500000	0.000000002	-0.52359877	$3.1x10^{-7}$

Hence $\xi = (0.5, 0.000000002, -0.52359877)^T$ .But the actual solution $p = (0.5, 0, \frac{-\pi}{6})^T$

Convergence Analysis of Fixed Point

Let us consider the system of equation

$$\begin{cases} f_1(x_1, \cdots, x_n) = 0 \\ f_2(x_1, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, \cdots, x_n) = 0 \end{cases} \quad \text{-----------------------------------} (3.5)$$

And suppose the system of equation have the solution

$$(\xi_1, \cdots, \xi_n)^T = \xi$$

Rewriting (3.5) in equivalent form

$$\begin{cases} x_1 = g_1(x_1,\cdots,x_n) \\ x_2 = g_2(x_1,\cdots,x_n) \\ \quad\vdots \\ x_n = g_n(x_1,\cdots,x_n) \end{cases} \text{------------------------------------}(3.6)$$

The iteration scheme can be written as:

$$\begin{cases} x_1^{(k+1)} = g_1(x_1^{(k)},\cdots,x_1^{(k)}) \\ \quad\vdots \\ x_n^{(k+1)} = g_n(x_1^{(k)},\cdots,x_n^{(k)}) \end{cases} \text{------------------------------------}(3.7)$$

And we have $(\xi_1,\cdots,\xi_n)^T = \xi$ is a fixed point of---------------------------------- (3.6)

$$\begin{cases} \xi_1 = g_1(\xi_1,\cdots,\xi_n) \\ \quad\vdots \\ \xi_n = g_n(\xi_1,\cdots,\xi_n) \end{cases} \text{--}(3.8)$$

Now subtracting (3.7) from (3.8) we have,

$$\begin{cases} \xi_1 - x_1^{(k+1)} = g_1(\xi_1,\cdots,\xi_n) - g_1(x_1^{(k)},\cdots,x_1^{(k)}) \\ \quad\vdots \\ \xi_n - x_n^{(k+1)} = g_n(\xi_1,\cdots,\xi_n) - g_n(x_1^{(k)},\cdots,x_n^{(k)}) \end{cases}$$

$$\Rightarrow \begin{cases} \xi_1 - x_1^{(k+1)} = g_1(\xi_1 - x_1^{(k)},\cdots,\xi_n - x_1^{(k)}) \\ \quad\vdots \\ \xi_n - x_n^{(k+1)} = g_n(\xi_1 - x_1^{(k)},\cdots,\xi_n - x_n^{(k)}) \end{cases}$$

And we have that

$$\begin{cases} \xi_1 - x_1^{(k+1)} = g_1((\xi_1 - x_1^{(k)}) + x_1^{(k)},\cdots,(\xi_n - x_n^{(k)}) + x_n^{(k)}) - g_1(x_1^{(k)},\cdots,x_n^{(k)}) \\ \quad\vdots \\ \xi_n - x_n^{(k+1)} = g_n((\xi_1 - x_1^{(k)}) + x_1^{(k)},\cdots,(\xi_n - x_n^{(k)}) + x_n^{(k)} - g_n(x^{(k)}{}_1,\cdots,x_n^{(k)}) \end{cases} \text{---------------}(3.9)$$

Since $\xi = x^{(k)} + e^{(k)}$ *and* $\xi_i = x_i^{(K)} + e_i^{(k)}$ Expanding (3.9) by Taylors series of several variable at $x^{(k)} = (x_1^{(K)},\cdots,x_n^{(K)})$ and neglecting the second and higher order derivative term to

$$\begin{cases} \xi_1 - x_1^{(k+1)} = g_1(x_1^{(k)}\cdots,x_n^{(k)}) + (\xi_1 - x_1^{(k)})\dfrac{\partial g_1}{\partial x_1^{(k)}} +,\cdots,+ (\xi_n - x_n^{(k)})\dfrac{\partial g_1}{\partial x_n^{(k)}}\ ) - g_1(x_1^{(k)},\cdots,x_n^{(k)}) \\ \vdots \\ \qquad\qquad\qquad\qquad \vdots \\ \xi_n - x_n^{(k+1)} = g_n(x_1^{(K)},\cdots,x_n^{(K)}) + ((\xi_1 - x_1^{(k)})\dfrac{\partial g_n}{\partial x_1^{(k)}} +,\cdots,+ (\xi_n - x_n^{(k)})\dfrac{\partial g_n}{\partial x_n^{(k)}} + -g_n(x^{(k)}{}_1,\cdots,x_n^{(k)}) \end{cases} \quad ---(3.10)$$

In matrix form, the error equation becomes

$$\begin{pmatrix} \xi_1 - x_1^{(k+1)} \\ \vdots \\ \xi_n - x_1^{(K+1)} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g_1(x^{(K)})}{\partial x_1^{(k)}} & \cdots & \dfrac{\partial g_1(x^{(k)})}{\partial x_n^{(K)}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n(x^{(K)})}{\partial x_1^{(K)}} & \cdots & \dfrac{\partial g_n(x^{(K)})}{\partial x_1^{(K)}} \end{pmatrix} \begin{pmatrix} \xi_1 - x_1^{(k)} \\ \vdots \\ \xi_n - x_n^{(k)} \end{pmatrix} --------------(3.11)$$

$$Let\ \ J(X^{(k)}) = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{pmatrix}$$

Which is the Jacobean matrix of the function $g_1,\cdots,g_n\ \ at\ \ X^{(k)}$**.**

Let $w_{k+1} = (x_1^{(k+1)},\cdots,x_n^{(k+!)})^T, W_K = (x_1^{(k)},\cdots,x_n^{(k)})^T$,then we can write

$$\xi - w_{k+1} = J(x^{(k)})(\xi - w_k) --------------------(3.12)$$

In (3.12),if w_k is closed to ξ ,then $J(x^{(k)})$ will closed to $J(\xi)$.

This will make the size of $J(\xi)$ crucial in analyzing the convergent in (3.12) and it plays in the role of $g'(\xi)$ in the theory of chapter -2.To measure the size of the error $\xi - w_k$ and matrix $J(x^{(k)})$ we will use the vector and matrix norms $\|.\|_\infty$

Returning to (3.12) we have

Hence the convergent of fixed point is linear and the iteration for two unknown converges when $\|J(X^{(k)}\|_\infty < 1$ for each iteration.

In order to convergences to be rapid enough to make the method advisable in any given it is necessarily that quantity $\|J\|_\infty$ be much less than 1

Newton's Methods

We extend the method derived for single equation $f(x)=0$ to a system of nonlinear equation.

Consider a system of n-nonlinear equation in n-unknown

$$\begin{cases} f_1(x_1,\cdots,x_n)=0 \\ f_2(x_1,\cdots,x_n)=0 \\ \quad\vdots \\ f_n(x_1,\cdots,x_n)=0 \end{cases} \text{-------------------------------------}(3.13)$$

Regarding the arguments $x_1,\cdots,x_n$ **as n-dimensional vector** $x=(x_1,\cdots,x_n)^t$ **.The entries**

$f_1,\cdots,f_n$ As n-dimensional vector function and $F(x)=(f_1,\cdots,f_n)^T$ .We can write (3.13) compactly as

$$F(x)=0 \text{--------------------------------}(3.14)$$

Let $X^{(k)}=(x_1^{(k)},\cdots,x_n^{(k)})^T$ *is the* k^{th} approximation of the root $x=(x_1,\cdots,x_n)^T$ of the vector equation can be represented as

$$x=x^{(k)}+e^{(k)} \text{------------------------------}(3.15)$$

Where $e^{(k)}=(e_1,\cdots,e_n)^T$ is the correction or the error of the root

From (3.14) and (3.15),we have that ;

$$F(x^{(k)}+e^{(k)})=0 \text{--------------------------------}(3.16)$$

Our aim is to find the value of $e^{(k)}$ using the k^{th} approximation $X^{(k)}$ so that $(k+1)^{th}$ approximation is better than the k^{th} approximation.

Assuming $F(x)$ is continuously differentiable in the domain containing x *and* $x^{(k)}$ and expanding each $f_i(x^{(k)}+e^{(k)})$ using Taylor's expansion for function of several variable about $X^{(k)}$.

$$f_1(x_1^{(k)}+e_1^{(k)}+\cdots+x_n^{(k)}+e_n^{(K)})=f_1(x_1,\cdots,x_n)+\left[\frac{\partial f_1(x^{(k)})}{\partial x_1^{(K)}}e_1^{(K)}+\cdots+\frac{\partial f_1(x^{(k)})}{\partial x_n^{(k)}}\right]+\text{higher order}$$

In the same way expand $f_2(x^{(k)}+e^{(k)}),\cdots upto\ f_n(x^{(k)}+e^{(k)})$.

Since $e_i^{(k)}$ are relatively small number, neglecting square and higher power of $e_i^{(k)}$ we obtain a system of linear equation as follow

$$\begin{cases} \frac{\partial f_1(x^{(k)})}{\partial x_1^{(k)}} e_1^{(K)} + \cdots + \frac{\partial f_n(x^{(k)})}{\partial x_n^{(K)}} = -f_1(x_1^{(k)}, \cdots, x_n^{(k)}) \\ \vdots \\ \frac{\partial f_n(x^{(k)})}{\partial x_1^{(k)}} e_1^{(k)} + \cdots + \frac{\partial f_n(x^{(k)})}{\partial x_n^{(k)}} = -f_n(x_1^{(k)}, \cdots, x_n^{(k)}) \end{cases} \text{-------------}(3.17)$$

More compactly can be written as:

$$F(x^{(k)}) + F'(x^{(k)})(e^{(k)}) = 0 \text{-------------------------}(3.18)$$

Where $F'(x^{(k)})$ **can be considered as the Jacobean Matrix and it is denoted by**

$$J(X^{(k)}) = \begin{pmatrix} \frac{\partial f_1(x^{(k)})}{\partial x_1^{(k)}} & \cdots & \frac{\partial f_1(x^{(k)})}{\partial x_n^{(k)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x^{(k)})}{\partial x_1^{(k)}} & \cdots & \frac{\partial f_n(x^{(k)})}{\partial x_n^{(k)}} \end{pmatrix} = \left(\frac{\partial f_i(x^{(k)})}{\partial x_j^{(k)}} \right), i, j = 1,2,\cdots,n$$

And $F(X^{(K)} = [f_1(x_1^{(k)}, \cdots, x_n^{(k)}), \cdots, f_n(x_1^{(k)}, \cdots, x_n^{(k)})]^T$.

Assuming the matrix $J(X^{(K)})$ **is non singular from (3.18) it follows**

$$e^{(k)} = J^{-1}(X^{(k)})F(X^{(K)}) \text{----------------------}(3.19)$$

Which is the value of the correction interims of $x^{(k)}$**.Since** $e^{(k)} = x^{(k+1)} - x^{(k)}$**.Then (3.19) becomes**

$$x^{(k+1)} = x^{(k)} - J^{-1}(x^{(k)})F(x^{(k)}) \text{--} (3.20)$$

Equation (3.20) is called Newton's method for nonlinear system of equation.

A definite weakness in the Newton's method procedure arises from the necessity of inverting the matrix J(X) at each step. In practice, the methods is performed in a two step manner, first a vector Y

Is found which satisfy

$$J(X^{(k)})Y = -F(X^{(k)}) \text{----------------------}(3.21)$$

After this has been accomplished, the new approximation $x^{(k+1)}$ **can be obtained by adding to** $x^{(k)}$**.**

In equation (3.21) we solve for Y by linear system of equation by direct methods if the system is small and by iterative methods if the system is large for each iteration. The convergence of Newton's method depends on the initial approximation $x^{(0)}$

Example: a) Take one step from a suitable starting point with Newtons method applied to the system

$$\begin{aligned} 10x+\sin(x+y) &= 1 \\ 8y-\cos^2(z-y) &= 1 \\ 12z+\sin z &= 1 \end{aligned}$$

Suggest for fixed point method $x^{(k+1)} = G(x^{(k)})$ and how many iteration are required to obtain a solution correct to six decimal point from starting point (a)

Solution: we have the system of equations

$$\begin{aligned} f_1(x,y,z) &= 10x+\sin(x-y)-1=0 \\ f_2(x,y,z) &= 8y-\cos^2(z-y)-1=1 \\ f_3(x,y,z) &= 12z+\sin z-1=0 \end{aligned}$$

To obtain a suitable starting point, we use the approximation

$$\begin{aligned} \sin(x+y) &\approx 0 \\ \cos(z-y) &\approx 1 \\ \sin z &\approx 0 \end{aligned}$$

And obtain the given initial approximation

$$x_0 = \frac{1}{10}, \quad y_0 = \frac{1}{4}, \quad z_0 = \frac{1}{12}$$

We have

$$J_k(x) = \begin{pmatrix} 10+\cos(x+y) & \cos(x+y) & 0 \\ 0 & 8-\sin(2(z-y)) & \sin(2(z-y)) \\ 0 & 0 & 12\cos z \end{pmatrix}$$

And $J_0 = \begin{pmatrix} 10.939373 & 0.939373 & 0 \\ 0 & 8.327195 & -0.327195 \\ 0 & 0 & 12.996530 \end{pmatrix}$

$$J_0^{-1} = \begin{pmatrix} 0.091413 & -0.010312 & -0.000260 \\ 0 & 0.120089 & 0.003023 \\ 0 & 0 & 0.076944 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0.3442898 \\ 0.027522 \\ 0.083237 \end{pmatrix}$$

Using the Newton's method

$$x^{(k+1)} = x^{(k)} - J^{-1}(x^{(k)})F(x^{(k)})$$

We obtain k=0

$$X^{(1)} = X^{(0)} - J_0^{-1}F_0$$

$$Hence \quad x_1 = 0.0689, y_1 = 0.246443, z_1 = 0.076929$$

we can write a fixed –point methods in the form

$$X^{(k+1)} = \frac{1}{10}[1-\sin(x_k + y_k)] = g_1(x_k, y_k, z_k)$$
$$Y^{(k+1)} = \frac{1}{8}[1-\cos^2(z_k - y_k)] = g_2(x_k, y_k, z_k)$$
$$Z^{(K+1)} = \frac{1}{12}[1-\sin z_k] = g_3(x_k, y_k, z_k)$$

Starting with initial approximation $x^{(0)} = \left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right)^T$,we obtain the sequence of iteration.

$$x^{(1)} = (0.065710, 0.246560, 0.076397)^T$$
$$x^{(2)} = (0.069278, 0.246415, 0.076973)^T$$
$$x^{(3)} = (0.068952, 0.246445, 0.076929)^T$$
$$x^{(4)} = (0.068978, 0.2464442, 0.076929)^T$$
$$x^{(5)} = (0.068978, 0.246442, 0.076929)^T$$

Hence, the solution correct to six decimal place is obtained after five iteration.

Convergence of Newton's Method

Newton's method converges quadratic ally. When carring out this method the system converges quiet rapidly once the approximation is closed to the actual solution of the nonlinear system. This is seen as a advantage because Newton's method may require less iteration, compared to another with a lower rate of convergence to reach a solution. However, the system does not converge, this is indicator that an error in computations has occurred, or a solution may not exist.

In the following proof, we will prove that Newton's method does indeed converge quadratic ally

Proof of Newton's Method Quadratic Convergence

In order for Newton's method to converges quadratic ally, the initial vector $X^{(0)}$ must be sufficiently close to a solution of the system F=0, which is denoted by ξ.As well, the Jacobean matrix at must not be singular, that is $J(x)^{-1}$

must exist. The goal of this proof to show that

$$\frac{\left\| X^{(K+1)} - \xi \right\|}{\left\| X^{(k)} - \xi \right\|^2} = \lambda$$ Where λ denote any positive constant

We have

$$\left\| e^{(k+1)} \right\| = \left\| X^{(k+1)} - \xi \right\| = \left\| X^{(k)} - J^{-1}(X^{(K)})F(X^{(K)}) - \xi \right\|$$

$$= \left\| X^{(K)} - \xi - J^{-1}(X^{(K)})F(X^{(K)}) \right\|$$

We set $\left\| e^{(k)} \right\| = \left\| X^{(k)} - \xi \right\|$

$$\Rightarrow \left\| e^{(k+1)} \right\| = \left\| e^{(k)} - J^{-1}(X^{(K)})F(X^{(K)}) \right\| \qquad \textbf{(3.22)}$$

Next we define the second –order Taylors series as

$$F(X^{(K)}) \cong F(\xi) + J(x^{(k)})(e^{(k)}) + \frac{1}{2}(e^{(k)})^T H\,(e^{(K)}) \qquad \textbf{(3.23)}$$

Where $J(x^{(k)})$ is the Jacobean and H $\left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j}$ We then have to multiply each side of the Taylors series by J^{-1} which yields

$$J^{-1}F(X^{(K)}) \cong J^{-1}[F(\xi) + J(X^{(K)})e^{(K)} + \frac{1}{2}(e^{(k)})^T H\, e^{(k)}]$$

$$= e^{(k)} + \frac{1}{2} J^{-1}(e^{(k)})^T H\, e^{(k)} \text{-------------------------} (3.24)$$

Using (3.22) and (3.24), we obtain our last results such that

$$\left\| X^{(K+1)} - \xi \right\| = \left\| e^{(k+1)} \right\|$$

$$= \left\| \frac{1}{2} J^{-1}(e^{(k)})^T H(e^{(k)}) \right\|$$

$$\leq \frac{1}{2} \left\| J^{-1} \right\| \left\| H \right\| \left\| e^{(k)} \right\|^2$$

$$\Rightarrow \frac{\left\| e^{(K+1)} \right\|}{\left\| e^{(k)} \right\|^2} \leq \left\| J^{-1} \right\| \left\| H \right\| = \lambda > 0$$

This show that Newton's method converges quadratic ally

Advantages and Disadvantage of Newtons method

One of the advantage of Newtons method is that not too complicated in form and it can used to solve a varity of problems.The major disadvantage assocaited with Newtons method is that

J(x),as well as its inversion has,to be calculated for each iteration.Calculating both the Jacobian matrix and its inverse can be quite time consuming depending on the size your system.Another prolbem that we may be challanged with when using Newtons method is that it may fail to converge.If Newtons method fail to converge this will results in oscillation between points.

CONCLUSIONS

From this seminar,it is safe to say that numerical methods are a vital strand of mathematics.They area powerful tool in not only solving nonlinear algeberic and trancedental equations with one variable,but also system of nonlinear algeberic and transedental equations.Even equations or systems of equations that may look simlistic in form,may in fact need the use of numerical methods in order to be solved.In this paper,we only examined two methods,however,there are several other ones that we have yet to take a closer look at.

The main results of this paper can be highlighted in to two different areas:Convergence and the role of Newtons and fixed point iteration methods.With regards to convergence,we can summarize that a numerical method with a higher rate of convergence may reach the solution of a system in less iteration in comarsion to another method with a slwoer rate convergence.For example,Newtons method converges quadratically and fixed point iteration method converges linearly.The implication of this would be that given the exact nonlinear system of equations denoted by F,Newtons method would arrive at the solution of F=0 in less iteration compared to Fixed point iteration method.

After all the material examined in this seminar,we can conclude that numerical methods are key componet in the area of nonlinear mathematics

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