2-D Problem of Generalized Thermoelastic Medium with Voids under the Effect of Gravity: Comparison of Different Theories

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Abstract The purpose of this paper is to study the 2-D problem of generalized thermoelastic medium with voids under the effect of gravity within the framework of the Green-Lindsay, Lord–Shulman and classical coupled theories. The normal mode analysis is used in our problem. Numerical results with comparisons between theories are illustrated graphically. Comparisons are made between the three theories in the presence and absence of gravity and also with and without voids.

Keywords gravity; Lord–Shulman; Green-Lindsay; thermoelasticity; normal mode analysis, voids. UDC 537.6, 539.3

1. Introduction

The effect of mechanical and thermal distribution of an elastic body is studied within the framework of the theory of thermoelasticity. The generalized theory of thermoelasticity is one of the modified versions of classical uncoupled and coupled theory of thermoelasticity that has been developed in order to remove the paradox of physical impossible phenomena of infinite velocity of thermal signals in the classical coupled thermoelasticity theory stated by Biot [1]. The coupled CD theory of thermo-elasticity was extended by including the thermal relaxation times into the constitutive equation by Lord and Shulman [2] and Green and Lindsay [3]. These theories eliminate the paradox of an infinite velocity of the heat propagation and were termed the generalization theories of thermoelasticity, there are the following differences between the two theories:

i. The Lord-Shulman L-S theory involves one relaxation time of the thermo-elastic \( \tau_0 \), while the Green and Lindsay G-L theory takes into account two relaxation times \( \tau_0, \tau_0, \tau_1 \).

ii. In the L-S theory, the energy equation involves the first and second derivatives of the strain with respect to time, whereas the corresponding equation in the G-L theory needs only the first derivative of this strain with respect to time.

iii. In the linear case, according to the approach of the G-L theory, the heat cannot propagate with a finite speed unless the stresses depend on the temperature, velocity, whereas according to the L-S theory, the heat can propagate with a finite speed even though the stresses are independent of the temperature, velocity.

iv. The two theories are structurally different from one another, and one cannot be obtained as a particular case of the other.

Theory of linear elastic materials with voids is an important development of the classical theory of elasticity; this theory deals with materials which have a distribution of small voids, where the volume of void is included among the kinematics variables and investigate various types of geological and biological materials since the classical theory of elasticity is not sufficient. The theory reduces to the classical theory in the limiting case of

In the present work, we have formulated the generalized thermoelastic medium with voids for three theories under the influence of gravity and solve for the components of displacement, stresses, temperature distribution. The normal mode method was used to obtain the exact expression for the considered variables. Comparisons are carried out between the considered variables as calculated from the generalized thermoelastic pours medium based on the L-S, G-L and CD theories in the absence and presence of gravity. In addition, a comparison made between the three theories with and without voids.

2. Formulation of the Problem and Basic Equations

We consider a homogeneous isotropic elastic body with voids in a half-space \( z \geq 0 \) under the effect of a constant gravitational field of acceleration \( g \). We are interested in plane strain in the \( xz \)-plane with displacement components \( u_1, u_3 \) such that \( u_1 = u_1(x, z, t), \ u_3 = u_3(x, z, t) \).

Case I:
The basic governing equations of a linear thermoelastic medium with voids under the effect of gravity based on the L-S, G-L and CD theories are

The stress-strain relation written as:

\[
\sigma_{ij} = \left[ \lambda e_{kk} + b \phi - \beta \left( 1 + \nu \frac{\partial}{\partial t} \right) \delta_{ij} + \mu e_{ij} \right],
\]

(1)

where \( \lambda \) and \( \mu \) are the Lame' constants, \( \delta_{ij} \) is the Kronecker delta,

\[
e_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right),
\]

(2)

The dynamical equations of an elastic medium are given by

\[
\mu \nabla^2 u_1 + \left( \lambda + \mu \right) \frac{\partial e}{\partial x} + b \frac{\partial \phi}{\partial x} - \beta (1 + \nu) \frac{\partial}{\partial t} \nabla T + \rho \frac{\partial u_3}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2},
\]

(3)

\[
\mu \nabla^2 u_3 + \left( \lambda + \mu \right) \frac{\partial e}{\partial z} + b \frac{\partial \phi}{\partial z} - \beta (1 + \nu) \frac{\partial}{\partial t} \nabla T - \rho \frac{\partial u_1}{\partial x} = \rho \frac{\partial^2 u_3}{\partial t^2},
\]

(4)

The equation of voids is

\[
\alpha \nabla^2 \phi - b e - \zeta \phi - \alpha_0 b \frac{\partial \phi}{\partial t} + m (1 + \nu) \frac{\partial}{\partial t} T = \rho \chi \frac{\partial^2 \phi}{\partial t^2},
\]

(5)

The heat conduction equation,

\[
K \nabla^2 T = \rho C_e \left( 1 + \tau_0 \frac{\partial}{\partial t} \right) \frac{T_1}{T_0} + \beta T_0 (1 + n_0 \tau_0 \frac{\partial}{\partial t}) \frac{\phi}{T_0} + m T_0 (1 + n_0 \tau_0 \frac{\partial}{\partial t}) \phi.
\]

(6)

Where, \( \sigma_{ij} \) are the components of stress tensor, \( e_{ij} \) are the components of strain, \( \lambda, \mu \) are the Lame' constants, \( \beta = (3\lambda + 2\mu)\alpha_t \) such that \( \alpha_t \) is the coefficient of thermal expansion, \( \delta_{ij} \) is the Kronecker delta.
\(\alpha, b, \xi, \omega_0, m, \chi\) are the material constants due to the presence of voids, \(\rho\) is the density, \(C_E\) is the specific heat at constant strain, \(n_0\) is a parameter, \(\tau_0, \nu\) are the thermal relaxation times, \(K\) is the thermal conductivity, \(T_0\) is the reference temperature is chosen so that \(|(T - T_0)/T_0| < 1\) \(\phi\) is the change in the volume fraction field.

For a two dimensional problem in \(xz\)-plane, Eq. (1) can be written as:

\[
\sigma_{ij} = \left[ \lambda \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z} \right) + b \delta_{ij} - \beta (1 + \nu) \frac{\partial}{\partial t} \right] \delta_{ij} + \mu \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial Z} \right), \quad i, j = 1, 3. 
\]  

(7)

For the purpose of numerical evaluation, we introduce dimensions variables

\[
(x', z') = \frac{a_0}{c_0} (x, z), \quad (u_1', u_3') = \frac{a_0}{c_1} (u_1, u_3), \quad \sigma_{ij}' = \frac{\sigma_{ij}}{c_1^2}, \quad \phi' = \frac{\phi}{c_1^2}, \quad \delta_{ij}' = \frac{\delta_{ij}}{c_1}, \quad T' = T/T_0, \quad t' = \phi t, \quad g' = \frac{g}{c_1 a_0}.
\]

\( \nu' = a_0^2 \nu, \quad \tau_0' = \phi \tau_0, \quad c_1^2 = \lambda + 2 \mu \rho, \quad a_0^2 = \frac{\rho \sigma_0}{c_1^2} \).\n
Using the above dimensions quantities, Eqs. (3)-(6) become

\[
\nabla^2 u_1 + A_1 \frac{\partial \phi}{\partial x} + A_2 \frac{\partial \phi}{\partial x} + \sigma_{ij} u_1 = A_5 \frac{\partial^2 u_1}{\partial t^2},
\]

(8)

\[
\nabla^2 u_3 + A_1 \frac{\partial \phi}{\partial z} + A_2 \frac{\partial \phi}{\partial z} - A_3 (1 + \nu) \frac{\partial T}{\partial t} + A_4 \frac{\partial u_1}{\partial x} = A_5 \frac{\partial^2 u_3}{\partial t^2},
\]

(9)

\[
\nabla^2 \phi - A_{11} \phi - A_{2} \frac{\partial \phi}{\partial x} + A_{3} (1 + \nu) \frac{\partial T}{\partial z} + A_{4} \frac{\partial u_1}{\partial x} = A_{5} \frac{\partial^2 \phi}{\partial t^2},
\]

(10)

\[
\epsilon_1 \nabla^2 t - A_{11} (1 + n_0 \tau_0) \frac{\partial T}{\partial t} + \epsilon_2 (1 + n_0 \tau_0) \frac{\partial T}{\partial t} = \frac{\sigma_0}{\rho \sigma_0} \frac{\partial \phi}{\partial t}.
\]

(11)

where, \(A_1 = \frac{\lambda + \mu}{\mu}, \quad A_2 = \frac{bc_1^2}{\mu a_0^2 \chi}, \quad A_3 = \frac{\beta T_0}{\mu}, \quad A_4 = \frac{\rho c_1^2}{\mu}, \quad A_5 = \frac{\rho c_1^2}{\mu}, \quad A_6 = \frac{\phi_0}{\chi}, \quad A_7 = \frac{\xi c_1^2}{\mu a_0^2 \chi}, \quad A_8 = \frac{a_0^2 \sigma_0}{\mu c_1^2}, \quad A_9 = \frac{m T_0 \chi}{c_1^2}, \quad A_{10} = \frac{\rho c_1^2}{\mu}, \quad A_{11} = \frac{\phi}{\mu c_1^2} \).

We define displacement potentials \(R\) and \(Q\) which relate to displacement components

\(u_1\) and \(u_3\) as

\[
u = \nabla^2 R, \quad u_3 = \frac{\partial R}{\partial x} - \frac{\partial Q}{\partial x},
\]

(12)

\[
u = \nabla^2 Q. \quad (\frac{\partial u_1}{\partial x} - \frac{\partial u_1}{\partial x}) = \nabla^2 Q.
\]

(13)

Using Eq. (13) in Eqs. (8)-(11), we obtain:

\[
(S_1 \nabla^2 - A_5 \frac{\partial^2}{\partial t^2}) R - A_4 \frac{\partial \phi}{\partial x} + A_2 \phi - A_3 (1 + \nu) \frac{\partial T}{\partial t} = 0,
\]

(14)

\[
A_4 \frac{\partial R}{\partial x} + (\nabla^2 - A_2 \frac{\partial^2}{\partial t^2}) Q = 0,
\]

(15)

\[
-A_6 \nabla^2 R + (\nabla^2 - A_7 - A_8 \frac{\partial^2}{\partial t^2}) \phi + A_9 (1 + \nu) \frac{\partial T}{\partial t} = 0,
\]

(16)

\[
\epsilon_2 \frac{\partial^2}{\partial t^2} \nabla^2 R - A_{11} (1 + n_0 \tau_0) \frac{\partial T}{\partial t} + \epsilon_1 \nabla^2 T - (1 + n_0 \tau_0) \frac{\partial T}{\partial t}.
\]

(17)

The components of stress tensor are
\[ \sigma_{xx} = A_{12} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial z^2} \right) + 2 \frac{\partial u_1}{\partial x} + A_{13} \phi - A_{14} T, \]  

(18)

\[ \sigma_{zz} = A_{12} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial z^2} \right) + 2 \frac{\partial u_1}{\partial z} + A_{13} \phi - A_{14} T, \]  

(19)

\[ \sigma_{xz} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial z}. \]  

(20)

Where \( A_{12} = \frac{-\lambda}{\mu}, \quad A_{13} = \frac{bc^2}{\mu\omega^2}, \quad A_{14} = \frac{\beta F_0}{\mu} (1 + \nu \omega), \quad S_1 = 1 + A_1. \)

### 3. Normal Mode Analysis

The solution of the considered physical variable decomposed in terms of normal modes as the following form

\[ \{R, Q, \phi, T, \sigma_{ij}\} (x, z, t) = \{R^*, Q^*, \phi^*, T^*, \sigma_{ij}^*\} (z) \exp[\pm(\omega t + cx)]. \]  

(21)

Where \( R^*, Q^*, \phi^*, T^*, \sigma_{ij}^* \) are the amplitudes of the functions \( R, Q, \phi, T, \sigma_{ij} \), \( \omega \) is the complex time constant, \( \nu = \sqrt{-1} \) and \( c \) is the wave number in the \( x \)-direction.

Using (21) in Eqs. (14)-(17), we obtain

\[ (D^2 - S_2) R^* - S_2 Q^* + S_4 \phi^* - S_2 T^* = 0, \]  

(22)

\[ S_6 R^* + (D^2 - S_7) Q^* = 0, \]  

(23)

\[ -A_6 (D^2 - c^2) R^* + (D^2 - S_8) \phi^* + S_9 T^* = 0, \]  

(24)

\[ S_{10} (D^2 - c^2) R^* - S_{11} \phi^* + (D^2 - S_{12}) T^* = 0. \]  

(25)

Where, \( S_2 = \frac{S_2 c^2 - A_3 \omega^2}{S_1}, \quad S_3 = \frac{iA_4 c}{S_1}, \quad S_4 = \frac{A_5}{S_1}, \quad S_5 = \frac{A_5 (1 + i \nu \omega)}{S_1}, \quad S_6 = iA_6, \quad S_7 = c^2 - A_4 \omega^2, \)

\[ S_8 = c^2 + A_7 + iA_4 \omega - A_{10} \omega^2, \quad S_9 = A_3 (1 + i \nu \omega), \quad S_{10} = \frac{-\epsilon_0 (i \omega + \eta_0 \sqrt{T_0} \omega^2)}{\epsilon_1}, \quad S_{11} = \frac{A_{11}}{\epsilon_1}, \quad S_{12} = \frac{c^2 + i \omega - \tau_0 \omega^2}{\epsilon_1}. \]

Eliminating \( Q^*, \phi^* \) and \( T^* \) between Eqs. (22)-(25), we get the following ordinary differential equation of the eighth order which satisfied with \( R^* \):

\[ [D^8 - B_1 D^6 + B_2 D^4 - B_3 D^2 + B_4] R^* (z) = 0. \]  

(26)

Where \( B_1 = S_{12} + S_8 + S_7 + S_2 - A_5 S_4 - S_9 S_{10}, \)

\[ B_2 = S_9 S_{12} + S_9 S_{11} + S_5 S_{12} + S_5 S_8 + S_2 S_{12} + S_2 S_8 + S_2 S_7 + S_5 S_3 - A_6 S_4 S_{12} - A_6 S_4 \omega^2 + S_4 S_9 S_{10}, \]

\[ -A_5 S_4 S_7 - A_5 S_5 S_{11} - S_5 S_8 S_{10} - S_5 S_8 S_{10}, \]

\[ B_3 = S_5 S_{12} + S_5 S_{11} + S_5 S_8 S_{10} + S_5 S_{12} + S_5 S_8 S_{10} + S_5 S_8 S_{12} + S_5 S_8 S_{10} + S_5 S_8 S_{12} + S_5 S_8 S_{10}, \]

\[ -A_5 S_5 S_7 - A_5 S_4 \omega^2 + S_4 S_7 S_{10} - A_6 S_4 S_{10} S_{12} - A_6 S_4 \omega^2 + S_4 S_9 S_{10}, \]

\[ B_4 = S_2 S_7 S_{12} + S_2 S_7 S_{11} + S_5 S_9 S_{12} + S_5 S_9 S_{11} + S_5 S_9 S_{12} + S_5 S_9 S_{10} - A_6 S_5 S_7 \omega^2 + S_5 S_9 S_{10}, \]

\[ -A_5 S_5 S_7 - S_5 S_7 S_{10} - S_5 S_7 S_{10}. \]

In a similar manner, we get

\[ [D^8 - B_1 D^6 + B_2 D^4 - B_3 D^2 + B_4] [R^* (z), Q^* (z), \phi^* (z), T^* (z)] = 0. \]  

(27)

Equation (27) factored as

\[ [(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)(D^2 - k_4^2)][R^* (z), Q^* (z), \phi^* (z), T^* (z)] = 0. \]  

(28)

Where \( k_n^2 (n = 1, 2, 3, 4) \) are the roots of the Eq. (27), \( D = \frac{d}{dz} \).

The solution of Eq. (27) bound as \( z \to \infty \), is given by:

\[ \text{Journal of Scientific and Engineering Research} \]
\[ R^* = \sum_{n=1}^{4} M_n e^{-k_n z}, \]  
\[ Q^* = \sum_{n=1}^{4} H_{1n} M_n e^{-k_n z}, \]  
\[ \phi^* = \sum_{n=1}^{4} H_{2n} M_n e^{-k_n z}, \]  
\[ T^* = \sum_{n=1}^{4} H_{3n} M_n e^{-k_n z}. \]  

Where, \( M_n (n=1,2,3,4) \) are constants.

To obtain the components of the displacement vector, from (29) and (33) in (12)
\[ u_1^* = \sum_{n=1}^{4} H_{4n} M_n e^{-k_n z}, \]  
\[ u_3^* = \sum_{n=1}^{4} H_{5n} M_n e^{-k_n z}, \]  
\[ u_5^* = \sum_{n=1}^{4} H_{7n} M_n e^{-k_n z}, \]  
\[ u_8^* = \sum_{n=1}^{4} H_{8n} M_n e^{-k_n z}. \]  

From Eqs. (31)-(34) in (18)-(20) to obtain the components of the stresses
\[ \sigma_{xx}^* = \sum_{n=1}^{4} H_{6n} M_n e^{-k_n z}, \]  
\[ \sigma_{zz}^* = \sum_{n=1}^{4} H_{7n} M_n e^{-k_n z}, \]  
\[ \sigma_{xz}^* = \sum_{n=1}^{4} H_{8n} M_n e^{-k_n z}. \]  

Where \( H_{1n} = \frac{-S_6}{(k_n^2 - S_7)} \), \( H_{2n} = \frac{-[A_0 S_5 (k_n^2 - c^2) + S_4 (k_n^2 - S_3 - S_4 H_{1n})]}{[S_5 (k_n^2 - S_7) + S_4 S_9]}, \) \( H_{3n} = \frac{k_n^2 - S_3 - S_4 H_{1n} + S_4 H_{2n}}{S_5}, \) \( H_{4n} = i c - k_n H_{1n}, \) \( H_{5n} = -(k_n + i c H_{1n}), \) \( H_{6n} = A_{12} (i c H_{4n} - k_n H_{5n}) + 2 i c H_{4n} + A_{13} H_{2n} - A_{14} H_{3n}, \) \( H_{7n} = A_{12} (i c H_{4n} - k_n H_{5n}) - 2 k_n H_{5n} + A_{13} H_{2n} - A_{14} H_{3n}, \) \( H_{8n} = -(k_n H_{4n} + i c H_{5n}). \)

4. Boundary Conditions
In this section, we need to consider the boundary conditions at \( z = 0 \), in order to determine the constants \( M_n (n=1,2,3,4) \).

(1) The mechanical boundary condition
\[ \sigma_{zz} = -P e^{i(\omega t + cx)}, \]  
\[ \sigma_{xz} = 0, \]  
\[ \frac{\partial \phi}{\partial z} = 0. \]  

(2) The thermal boundary condition that the surface of the half-space is subjected to
\[ T = P_2 e^{i(\omega t + cx)}. \]  

Where the magnitude of the applied force in the half-space is \( P_1 \), and \( P_2 \) is the applied constant temperature to the boundary.

Using the expressions of the variables into the above boundary conditions (38), (39), we obtain
\[ \sum_{n=1}^{4} H_{7n} M_n = -P_1, \]  
\[ \sum_{n=1}^{4} H_{8n} M_n = 0, \]
\[\sum_{n=1}^{4} - k_n H_{2n} M_n = 0, \quad (42)\]
\[\sum_{n=1}^{4} H_{3n} M_n = P_2. \quad (43)\]
Invoking boundary conditions (40)-(43) at the surface \(z = 0\) of the plate, we obtain a system of four equations. After applying the inverse of matrix method, we get the values of the four constants \(M_n\) (\(n = 1, 2, 3, 4\)).

**Case 2:**
The solution of wave propagation of a generalized thermoelastic medium without voids under the effect of gravity is: by putting \(\alpha, \beta, \xi, \alpha_0, \mu, \chi\) equal to zero, then the basic governing Eqs. (1) and (3)-(6) of a linear thermoelastic medium without voids under the effect of gravity can be written as:

\[
\mu \nabla^2 u_1 + (\lambda + \mu) \frac{\partial e}{\partial x} - \beta (1 + \nu) \frac{\partial T}{\partial t} \frac{\partial e}{\partial x} + \rho g \frac{\partial u_3}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2},
\]

\[
\mu \nabla^2 u_3 + (\lambda + \mu) \frac{\partial e}{\partial z} - \beta (1 + \nu) \frac{\partial T}{\partial t} \frac{\partial e}{\partial z} - \rho g \frac{\partial u_1}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2},
\]

\[
K \nabla^2 T = \rho C_E (1 + \tau_0) \frac{\partial T}{\partial t} + \beta T_0 (1 + n_0 \tau_0) \frac{\partial e}{\partial t},
\]

\[
\sigma_{ij} = [(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}) - \beta (1 + \nu) \frac{\partial T}{\partial t}) \delta_{ij} + \mu (\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x})], \quad i,j = 1,3.
\]

The dimensions of Eqs. (44)-(46) have the form

\[
E_1 \frac{\partial^2 e}{\partial x^2} - E_2 (1 + \nu) \frac{\partial e}{\partial x} \frac{\partial T}{\partial x} + E_3 \frac{\partial u_3}{\partial x} = E_4 \frac{\partial^2 u_1}{\partial t^2},
\]

\[
E_2 \frac{\partial e}{\partial z} - E_3 (1 + \nu) \frac{\partial e}{\partial z} \frac{\partial T}{\partial z} - E_4 \frac{\partial u_1}{\partial z} = E_4 \frac{\partial^2 u_3}{\partial t^2},
\]

\[
\epsilon_3 \nabla^2 T = (1 + \tau_0) \frac{\partial T}{\partial t} + \epsilon_4 (1 + n_0 \tau_0) \frac{\partial e}{\partial t}.
\]

Where \(E_1 = \frac{\lambda + \mu}{\mu}, \quad E_2 = \frac{\beta T_0}{\mu}, \quad E_3 = \frac{\rho g e^2}{\mu}, \quad E_4 = \frac{\rho e^2}{\mu}, \quad \epsilon_3 = \frac{K o^2}{\rho e^2}, \quad \epsilon_4 = \frac{\beta}{\rho e} \).

Using Eq. (13) in Eqs. (48)-(50), we obtain:

\[
(F_1 \nabla^2 - E_4 \frac{\partial^2}{\partial t^2})R - E_3 \frac{\partial}{\partial x} Q - E_2 (1 + \nu) \frac{\partial T}{\partial t} = 0,
\]

\[
E_3 \frac{\partial}{\partial x} R + (\nabla^2 - E_4 \frac{\partial^2}{\partial t^2})Q = 0,
\]

\[- \epsilon_3 \frac{\partial^2}{\partial t^2} \nabla^2 R + \epsilon_4 \nabla^2 T - (1 + \tau_0) \frac{\partial T}{\partial t} = 0.
\]

The components of stress tensor are

\[
\sigma_{xx} = E_3 [\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}] + 2 \frac{\partial u_1}{\partial x} - E_6 (1 + i \nu \omega) T,
\]

\[
\sigma_{zz} = E_3 [\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}] + 2 \frac{\partial u_3}{\partial z} - E_6 (1 + i \nu \omega) T,
\]

\[
\sigma_{xz} = \left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right].
\]

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Where \( E_s = \frac{\lambda}{\mu} \), \( E_6 = \frac{\beta F_0}{\mu} (1 + i \omega \omega) \), \( F_1 = 1 + E_1 \)

Using (21) in Eqs. (51)-(53), we obtain

\[
(D^2 - F_2) R^* - F_3 Q^* - F_4 T^* = 0, \tag{57}
\]

\[
F_2 R^* + (D^2 - F_6) Q^* = 0, \tag{58}
\]

\[
E_7 (D^2 - c^2) R^* + (D^2 - E_8) T^* = 0, \tag{59}
\]

where, \( F_2 = \frac{F_0^2 - E_4 \omega^2}{F_1} \), \( F_3 = \frac{i c E_3}{F_1} \), \( F_4 = \frac{E_2 (1 + i \omega \omega)}{F_1} \), \( F_5 = i c E_3 \), \( F_6 = c^2 - E_4 \omega^2 \),

\[
F_7 = -\frac{E_4 (i \omega - n_0 \tau_0 \omega^2)}{\varepsilon_3}, \quad F_8 = \frac{\varepsilon_3 c^2 + i \omega + \tau_0 \omega^2}{\varepsilon_3}.
\]

Eliminating \( Q^* \) and \( T^* \) between Eqs. (57)-(59), we get the following ordinary differential equation of sixth order which satisfied with \( R^* \)

\[
[D^6 - I_1 D^4 + I_2 D^2 - I_3] R^*(z) = 0. \tag{60}
\]

Where \( I_1 = F_8 + F_6 + F_2 - F_4 F_7 \), \( I_2 = F_6 F_8 + F_2 F_6 + F_2 F_6 + F_6 F_8 - F_4 F_6 F_7 - F_4 F_6^2 \), \( I_3 = F_2 F_6 F_8 + F_2 F_6 - F_4 F_6 F_7 c^2 \).

In a similar manner, we get

\[
[D^6 - I_1 D^4 + I_2 D^2 - I_3] [R^*(z), Q^*(z), T^*(z)] = 0. \tag{61}
\]

Equation (61) factored as

\[
[(D^2 - k_1^2)(D^2 - k_1^2^2)(D^2 - k_3^2)] [R^*(z), Q^*(z), T^*(z)] = 0. \tag{62}
\]

Where \( k_n^2 (n = 1, 2, 3) \) are the roots of the Eq. (61).

The solution of Eq. (61) bound as \( z \rightarrow \infty \), is given by:

\[
R^* = \sum_{n=1}^{3} M_n e^{-k_n z}, \tag{63}
\]

\[
Q^* = \sum_{n=1}^{3} L_{1n} M_n e^{-k_n z}, \tag{64}
\]

\[
T^* = \sum_{n=1}^{3} L_{2n} M_n e^{-k_n z}, \tag{65}
\]

where \( M_n (n = 1, 2, 3) \) are some constants.

To obtain the components of the displacement vector, from (63) and (64) in (12)

\[
u_1^* = \sum_{n=1}^{3} L_{3n} M_n e^{-k_n z}, \tag{66}
\]

\[
u_3^* = \sum_{n=1}^{3} L_{4n} M_n e^{-k_n z}. \tag{67}
\]

From Eqs. (63)-(65) in (54)-(56) to obtain the components of the stress vector.
\[ \sigma_{xx}^* = \sum_{n=1}^{3} L_{6n} M_n e^{-k_n z}, \quad (68) \]
\[ \sigma_{zz}^* = \sum_{n=1}^{3} H_{6n} M_n e^{-k_n z}, \quad (69) \]
\[ \sigma_{xz}^* = \sum_{n=1}^{3} H_{7n} M_n e^{-k_n z}, \quad (70) \]

Where
\[ L_{4n} = \frac{-F_4}{(k_n^2 - F_0)}, \quad L_{2n} = \frac{-F_2(k_n^2 - c^2)}{(k_n^2 - S_8)} , \quad L_{3n} = i c + k_n L_{4n}, \quad L_{4n} = -(k_n + i c L_{4n}), \]
\[ L_{5n} = E_5 (i c L_{3n} - k_n L_{4n}) + 2i c L_{3n} - E_6 L_{2n}, \quad H_{6n} = E_3 c L (-k_n L_{6n} - k L_n - E) L_n \]
\[ L_{7n} = -(k_n L_{3n} + i c L_{4n}). \]

5. Boundary Conditions

In this section, we need to consider the boundary conditions at \( z = 0 \), in order to determine the constants \( M_n (n = 1, 2, 3). \)

(1) The mechanical boundary conditions
\[ \sigma_{zz} = -P_1 e^{i(\alpha z + \chi z)}, \quad \sigma_{xz} = 0, \quad (71) \]

(2) The thermal boundary condition that the surface of the half-space subjected to
\[ T = P_2 e^{i(\alpha z + \chi z)}. \quad (72) \]

Where \( P_1 \) is the magnitude of the applied force in of the half-space and \( P_2 \) is the applied constant temperature to the boundary.

Using the expressions of the variables into the above boundary conditions (71), (72), we obtain
\[ \sum_{n=1}^{3} L_{6n} M_n = -P_1, \quad (73) \]
\[ \sum_{n=1}^{3} L_{7n} M_n = 0, \quad (74) \]
\[ \sum_{n=1}^{3} L_{2n} M_n = P_2. \quad (75) \]

Invoking boundary conditions (73)-(75) at the surface \( z = 0 \) of the plate, we obtain a system of three equations.

After applying the inverse of matrix method, we can get the values of the three constants \( M_n (n = 1, 2, 3). \)

6. Numerical results and discussion

In order to illustrate the obtained theoretical results in the preceding section, following Dhaliwal and Singh [15] the magnesium material chosen for purposes of numerical evaluations. The constants of the problem taken as
\[ \lambda = 2.14 \times 10^{10} \text{ N/m}^2, \quad \mu = 3.278 \times 10^{10} \text{ N/m}^2, \quad K = 1.7 \times 10^5 \text{ W/m deg}, \quad \alpha_t = 1.78 \times 10^{-5} \text{ N/m}^2, \quad T_0 = 298 \text{ K}, \]
\[ \rho = 1.74 \times 10^3 \text{ Kg/m}^3 , \quad C_K = 1.04 \times 10^3 \text{ J/Kg deg}, \quad \beta = 2.68 \times 10^6 \text{ N/m}^2 \text{ deg}, \quad \alpha_1^* = 3.58 \times 10^{13} \text{ /s}. \]

The voids parameters are
\[ \chi = 1.753 \times 10^{-15} \text{ m}^2, \quad \xi = 1.475 \times 10^{10} \text{ N/m}^2, \quad b = 1.13849 \times 10^{10} \text{ N/m}^2, \quad \alpha = 3.688 \times 10^{-5} \text{ N}, \]
\[ m = 2 \times 10^9 \text{ N/m}^2 \text{ deg}, \quad \alpha_0 = 0.0787 \times 10^{-3} \text{ N/m}^2 \text{s}. \]

The comparisons carried out for
$x = 0.5, \ t = 0.03, \ c = 0.2, \ \omega = \zeta_0 + i \zeta_1, \ \zeta_0 = -0.6, \ \zeta_1 = - \ p_1 = 0.1, \ p_2 = 2, \ \tau_0 = 0.05 \ s, \ \nu = 0.5 \ s, \ 0 \leq z \leq 7.$

The computations were carried out at $t = 0.03$. The numerical technique, outlined above, was used for the distribution of the real part of the displacement $u_3$, the stresses $\sigma_{zz}, \sigma_{xz}$ and the change in the volume fraction field $\phi$ with distance $z$, for the problem under consideration. All the considered variables depend not only on the variables $t, z$ and $x$, but also on the thermal relaxation times $\tau_0$ and $\nu$. The results are shown in Figs. 1-6. The graphs show the six curves predicted by three different theories of thermoelasticity (CD, L-S, and G-L). In these figures, the solid lines represent the solution in the CD theory, the dashed lines represent the solution with the G-L theory, and the dotted-dashed lines represent the solution with the L-S theory. Here, all the variables are taken in non-dimensional forms and we consider four cases:

1. Equations of the CD theory, when $n_0 = 0, \ \tau_0 = \nu = 0$.
2. Lord and Shulman L-S theory when $n_0 = 1, \ \nu = 0, \ \tau_0 > 0$.
3. Green and Lindsay G-L theory when $n_0 = 0, \ \nu > \tau_0 > 0$.
4. The three theories in the absence of a gravity field from the above mentioned by taking $g = 0$.

Figs. 1-3 show comparisons among the considered variables in the absence and presence of the gravity effect ($g = 0, \ g = 9.8$).

Fig. 1 shows that the distribution of the vertical displacement $u_3$ always begins from a positive value with gravity, in the context of the three theories CD, L-S, and G-L, it decreases in the range $0 \leq z \leq 1.5$, then increases in the range $4 \leq z \leq 7$, while increases in the range $1.5 \leq z \leq 4$. However, without gravity it begins from zero, and increases in the range $0 \leq z \leq 1.5$, then, decreases in the range $1.5 \leq z \leq 3.5$. Fig. 2 depicts the distribution of the change in the volume fraction field with and without gravity under the three theories. The values of $\phi$ with gravity are decreasing in the range $0 \leq z \leq 1.8, \ \phi$ then increasing in the range $1.8 \leq z \leq 4.5$. The values of $\phi$ without gravity is greater than that with gravity in the range $0 \leq z \leq 1.8, \ \phi$ while the vice versa in the range $1.8 \leq z \leq 6$. Fig. 3 shows that the distribution of the stress component $\sigma_{zz}$ begins from zero in the context of the three theories, and satisfies the boundary conditions at $z = 0$. The values of $\sigma_{zz}$, with gravity are greater than that without gravity.

Figs. 4-6 show comparisons among the considered variables in the absence and presence of voids in the case of material with gravity. Fig. 4 exhibits that the distribution of the vertical displacement $u_3$ begins from positive values in the presence and absence of voids. In the context of the three theories, we notice that the values of $u_3$ with voids are greater than that without voids in the range $0 \leq z \leq 1.5$, while the vice versa in the range $1.5 \leq z \leq 5.5$, then converges to zero. Fig. 5 determines the distribution of the stress component $\sigma_{zz}$, in the context the three theories in the presence and absence of voids. It explained that the distribution of $\sigma_{zz}$ increases with the increase of the values of voids in the range $0 \leq z \leq 2$, then decreases with the increase of the void in the range $2 \leq z \leq 6$. Fig. 6 demonstrates that the distribution of the stress component $\sigma_{zz}$, in the context of the three theories begins from zero and satisfies the boundary conditions at $z = 0$ with and without voids. It is noticed that the value of $\sigma_{zz}$ with voids is greater than that without voids in the range $0 \leq z \leq 3.7$, and the vice versa in the range $3.7 \leq z \leq 7$.

7. Conclusion

By comparing the figures that were obtained for the three thermoelastic theories, important phenomena are observed:

1. The values of all physical quantities converge to zero with increasing distance $z$, and all functions are continuous.
2. The normal mode analysis technique has been used is applied to a wide range of problems in thermodynamics and thermoelasticity.
3. All the physical quantities satisfy the boundary conditions.
4. The gravity has a significant effect on the variation of the considered physical quantities, since they make great changes in the behavior of the functions, also the same observation of the absence and presence of the voids in the thermoelastic solid.

Figure 1: The displacement component $u_3$ distribution against $z$ with and without gravity

Figure 2: The displacement of volume fraction field $\phi$ against $z$ with and without gravity

Figure 3: The displacement of the stress tensor $\sigma_{xz}$ against $z$ with and without gravity
Figure 4: The displacement component $u_3$ distribution against $z$ with and without voids.

Figure 5: The displacement of the stress tensor $\sigma_{zz}$ against $z$ with and without voids.

Figure 6: The displacement of the stress tensor $\sigma_{xz}$ against $z$ with and without voids.
References


