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Transient analysis of nonolocal microstretch thermoelastic thick circular plate with phase lags

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ABSTRACT

The present investigation is concerned with the eigenvalue approach in a homogeneous, isotropic nonlocal microstretch thermoelastic circular plate with three phase lag model subjected to thermomechanical sources. The components of displacements, microrotation, microstretch, temperature distribution, normal stress, shear stress and couple shear stress are obtained in the transformed domain by using the Laplace and Hankel transforms. The resulting quantities are obtained in the physical domain by applying numerical inversion technique. Effects of nonlocal, phase lag, relaxation time, with and without energy dissipation are analyzed on the resulting quantities numerically and illustrated graphically.

1. Introduction

The nonlocal elasticity theory of continuum mechanics has received more attention recently. The basic idea behind the nonlocal theories is that the interacting forces between material points are far-reaching in character. The nonlocal theories are differing from the local theories by taking into account the master balance laws which is valid only on the whole body. In many cases, such as in phonon dispersion in solid, in fracture mechanics, in surface physics, in electromagnetic solids, the nonlocal effects are dominant and only the nonlocal theories might provide the right answer, while local theories would fail. The nonlocal elasticity theory can be employed to determine the dispersion relations of elastic crystals, faithfully in the entire Brillouin zone. The physical phenomena involving microscopic internal characteristic length should be predictable by means of the nonlocal elasticity.

The nonlocal theory takes into account nonlocal effects hitherto almost entirely neglec-

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ted in the mechanics of continua. In this theory, the various physical quantities defined at a point not as a function of the values of independent constitutive variables at that point only but as a function of their values over the whole body. Eringen and Edelen (1972) and Eringen (1972) investigated a nonlocal theory of elastic solids. In this theory, the balance laws contain nonlocal residuals of fields and these residuals are determined with the constitutive equations which are the basis of certain invariance requirements and thermodynamic restrictions. The constitutive equations and the nonlocal residuals are functional of the motions and deformation gradients of all points of the body. A reciprocal theorem and two variational principles characterizing initial boundary value problems are presented by Altan (1990). Wang and Dhaliwal (1993) established a work and energy theorem and a uniqueness theorem without making any definiteness assumptions about the elastic moduli and also investigated a reciprocal theorem. Povstenko (1999) presented the nonlocal theory of elasticity and its applications to the description of defects in solid bodies.

Paola, Pirrotta and Zingales (2010) presented the variational formulation of the problem to describe the mathematical consistency of the proposed model of the linearly elastic problem and also discussed the virtual work theorem in the presence of long-range interactions chracterized by range dependent nonlocal interactions. Carpinteri, Cornetti and Sapora (2014) used the spatial fractional calculus to investigate a material whose nonlocal stress is defined as the fractional integral of the strain field. Pandey, Nashalm and Holm (2015) applied the framework of tempered fractional calculus to discuss the spatial dispersion of elastic waves in a one dimensional elastic bar. Koutsoumaris, Eptaimeros and Tsamasphyros (2017) formulated the nonlocal continuum theory, either integral or differential form which is widely used to explain size effect phenomena in

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micro and nano structures. They investigated the static response of a beam by making use of modified kernel and the kernel corresponding to the two phase nonlocal integral model with various types of loading and boundary conditions. Liew, Zhang and Zhang (2017) presented a literature review of recent research studies on the applications of nonlocal elasticity theory in the modeling and simulation of grapheme sheets.

Eringen (1974) derived the basic equations of nonlocal thermoelasticity. Balta and Suhubi (1977) also derived the constitutive equations of the heat conduction with the second order temperature rates. Inan and Eringen (1991) studied the wave propagation in thermoelastic plates within the context of nonlocal thermoelasticity theory. Wang and Dhaliwal (1993) derived the work and energy equation in a generalized nonlocal thermoelasticity and also proved that the initial boundary value problem has a unique solution. Ezzat and Youssef (2013) studied the influence of the Thomson heating and the Fourier's heat conduction on half space thermoelectricity solid in the presence of magnetic field subjected to a thermal shock. Zenkour and Abouelregal (2014) constructured a new model of nonlocal thermoelasticity beam theory with phase lags subjected to a harmonically varying heat considering the thermal conductivity to a variable. Yu, Tian and Xiong (2016) established a size dependent thermoelastic model for higher order simple material by adopting both the size effect of heat conduction and elasticity with the aids of extended irreversible thermodynamics and generalized free energy.

Roychoudhuri (2007) investigated a three phase lag model by taking the heat conduction law that includes temperature gradient and thermal displacement gradient among constitutive variables in the theory of coupled thermoelasticity. This model is an extension of the thermoelastic models proposed by Lord Shulman (1967) and Tzou (1995a, 1995b). Kumar and Chawla (2013) studied the propagation of longitudinal and transverse waves at the interface between uniform elastic solid half space and thermoelastic solid with three phase lag model. El-Karamany and Ezzat (2013) established the uniqueness and reciprocal theorems and variational principle for the three phase lag micropolar thermoelasticity theory. Othman, Hasona and Abd-Elaziz (2015) studied the effect of the rotation and initial stress on a two dimensional problem of micropolar thermoelastic isotropic medium with three phase lag theory. Zenkour (2016) presented the generalized thermoelasticity theory based on the dual phase lags theory to describe the problem of a thick walled simply supported beam with different applied heat source and mechanical loads. Kartoshov (2016) presented a mathematical theory for boundary value problems of nonstationary heat conduction by using dual phase lag and also presented the features of analytical solutions of such heat problems. Marin, Agarwal and Coarcea (2017) established a uniqueness and reciprocal theorem for a three phase lag dipolar thermoelastic body and also proved variational principle. Ezzat, El-Karamany and El-Bary (2017) presented a mathematical model of two temperature Green-Naghdi thermoelasticity theories based on fractional derivative heat transfer with phase lads. Othman and Mansour (2017) studied the deformation of thermoelastic half space under magnetic field, gravity, rotation and hydrostatic initial stress in the context of three phase theory of thermoelasticity.

In this paper, we investigated an eigenvalue approach for a homogeneous, isotropic nonlocal microstretch thermoelastic circular plate with three phase lag model subjected to thermomechanical sources. The expressions of components of displacement, microrotation, microstretch, temperature distribution, normal stress, shear stress and couple shear stress are obtained in the transformed domain by using the Laplace and Hankel transforms. These expressions are obtained in the physical domain by applying numerical inversion technique. We have depicted the effects of nonlocal, phase lag, relaxation time, with and without energy dissipation on the resulting expressions. The resulting quantities are presented numerically and illustrated graphically.

2. Basic equations

Following Eringen (1999, 2002) and Roychoudhuri (2007), the constitutive relations for nonlocal microstretch thermoelastic medium in the absence of body forces, body couples, heat sources and extrinsic equilibrated body force are taken as

$$\left(1-\varepsilon^2\nabla^2\right)t_{kl}=t_{kl}^C=\left[\lambda_0\psi\left(x\right)+\lambda\varepsilon_{rr}\left(x\right)\right]\delta_{kl}+\left(\mu+K\right)\varepsilon_{kl}\left(x\right)+\mu\varepsilon_{lk}\left(x\right)-\nu\nabla T \quad (1)$$

$$\left(1-\varepsilon^2\nabla^2\right)m_{kl}=m_{kl}^C=\left[b_0\varepsilon_{mlk}\psi_{,x}\left(x\right)+\alpha\gamma_{rr}\left(x\right)\right]\delta_{kl}+\beta\gamma_{kl}\left(x\right)+\gamma\gamma_{lk}\left(x\right) \quad (2)$$

$$\left(1-\varepsilon^2\nabla^2\right)\lambda_k=\lambda_k^C=\left[\alpha_0\psi_{,k}\left(x\right)+b_0\varepsilon_{klm}\gamma_{lk}\left(x\right)\right] \quad (3)$$

$$\left(1-\varepsilon^2\nabla^2\right)(s-t)=(s-t)^C=\lambda_1\psi(x)+\lambda_0\varepsilon_{kk}(x) \tag{4}$$

$$\rho T_0\dot{S}=-q_{,ii} \tag{5}$$

$$\rho T_0\dot{S}=\rho C_E\dot{T}(x)+\nu T_0\dot{\varepsilon}_{kk}(x)+mT_0\psi(x) \tag{6}$$

The quantities t_{kl}^C, m_{kl}^C, λ_k^C and $(s-t)^C$ are given by Eringen (1999) for classical local microstretch elastic solid.

Equations of motion for a nonlocal isotropic microstretch solid are given by Eringen (1984).

$$t_{kl,k}+\varepsilon_{imn}t_{mn}+\rho\left(l_l-j\ddot{u}_l=0\right) \tag{7}$$

$$m_{kl,k}+\varepsilon_{lmn}t_{mn}+\rho\left(l_l-j\ddot{\phi}_l\right)=0 \tag{8}$$

$$\lambda_{k,k}+(t-s)+\rho\left(l-\frac{1}{2}j_0\ddot{\psi}_l\right)=0 \tag{9}$$

$$\left(1+\eta_0\tau_0\frac{\partial}{\partial t}\right)q_i=-KT_{,i} \tag{10}$$

where f_l is the applied body force density, l_l is the body moment density and is applied scalar microstretch tensor.

Now using the constitutive relations (1)-(6) into the equations of motion (7)-(10), we obtain

$$(\lambda+\mu)\nabla(\nabla.\vec{u})+(\mu+K)\nabla^2\vec{u}+K\nabla\times\vec{\phi}+\lambda_0\nabla\psi\nabla T=\rho\left(1-\varepsilon^2\nabla^2\right)\frac{\partial^2\vec{u}}{\partial t^2} \tag{11}$$

$$(\alpha+\beta+\gamma)\nabla\left(\nabla.\vec{\phi}\right)-\gamma\nabla\times\left(\nabla\times\vec{\phi}\right)+K\nabla\times\vec{u}-2K\vec{\phi}=\rho j\left(1-\varepsilon^2\nabla^2\right)\frac{\partial^2\vec{\phi}}{\partial t^2} \tag{12}$$

$$\alpha_0\nabla^2\psi-\lambda_0(\nabla.\vec{u})-\lambda_1\psi+mT=\frac{1}{2}\rho j_0\left(1-\varepsilon^2\nabla^2\right)\frac{\partial^2\psi}{\partial t^2} \tag{13}$$

$$\begin{aligned}&\left[K^*\left(1+\tau_\nu\tfrac{\partial}{\partial t}\right)+K_1^*\tfrac{\partial}{\partial t}\left(1+\tau_t\tfrac{\partial}{\partial t}\right)\right]\nabla^2T=\\&\left(1+\tau_q\tfrac{\partial}{\partial t}+\tfrac{\tau_q^2}{2}\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial^2}{\partial t^2}\left[\rho C^*T+m\mathrm{T}_0\phi^*+\nu\mathrm{T}_0(\nabla.\vec{u})\right]\end{aligned} \tag{14}$$

$$\lambda_i^*=\alpha_0\psi_{,i}+b_0\varepsilon_{ijk}\phi_{j,k} \tag{15}$$

where $\lambda,\ \mu$ are Lame's constants, $\alpha,\ \beta,\ \gamma,\ K$ are constants of local micropolarity, $m,\ \alpha_0,\ b_0,\ \lambda_0,\ \lambda_1,\ j_0$ are constants of local microstretch elasticity, $\vec{u}$ is the displacement vector, $\vec{\phi}$ is the microrotation vector, ψ is the scalar microstretch, $\varepsilon = e_0 a$ is a nonlocal parameter, e_0 is a material constant a and being the internal characteristic length. The internal characteristic length a is the interatomic distance or lattice distance, $\nu = (3\lambda + 2\mu + K)\,\alpha_t$, α_t is the coefficient of linear thermal expansion, C^* is the specific heat at constant strain, K_1^* is the coefficient of thermal conductivity, τ_t, τ_q and τ_v the phase lag of the temperature gradient, the phase lag of the heat flux and the phase lag of the thermal displacement, T is the change in temperature of the medium at any time, t_{ij}, m_{ij} and δ_{ij} are the stress tensor, couple stress tensor and kroneckor delta and $\nabla^2 = \frac{\partial^2}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \mathrm{z}^2}$ is the Laplacian operator.

3. Formulation of the Problem

We consider a homogeneous and isotropic micropolar porous thermoelastic circular plate of thickness $2d$ occupying the region defined by $0 \le r \le \infty$, $-d \le z \le d$. We assume a two dimensional problem with cylindrical polar coordinate system $(r,\ \theta,\ z)$ having symmetry about z-axis. Therefore, all the quantities are independent of θ i.e. $\frac{\partial}{\partial \theta} = 0$. We assume the origin of the coordinate system $(r,\ \theta,\ z)$ at the middle surface of the plate and the z-axis is normal to it along the thickness. The initial temperature in the thick circular plate is taken as a constant temperature T_0. Since, we are considering a two dimensional problem, so, the components of displacement vector $\vec{u}$ and microrotation vector $\vec{\phi}$ are taken as

$$\vec{u} = (u_r,\quad 0,\quad u_z)\,, \qquad \vec{\phi} = (0,\quad \phi_\theta,\quad 0) \tag{16}$$

Introduce the following non-dimensional variables

$$\begin{aligned} r' &= \frac{\omega^* r}{c_1},\quad z' = \frac{\omega^* z}{c_1},\quad u_r' = \frac{\rho c_1 \omega^* u_r}{\nu T_0},\quad u_z' = \frac{\rho c_1 \omega^* u_z}{\nu T_0},\quad \phi_\theta' = \frac{\rho c_1^2 \phi_\theta}{\nu T_0},\ \psi' = \frac{\rho c_1^2 \psi}{\nu T_0} \\ T' &= \frac{T}{T_0},\ t' = \omega^* t,\quad \tau_t' = \omega^* \tau_t,\quad \tau_q' = \omega^* \tau_q,\quad t_{ij}' = \frac{t_{ij}}{\nu T_0},\quad m_{ij}' = \frac{\omega^*}{c_1 \nu T_0} m_{ij} \end{aligned} \tag{17}$$

where

$$c_1^2 = \frac{\lambda + 2\mu + K}{\rho},\ \omega^* = \frac{K}{\rho j}$$

Laplace and Hankel transforms are defined as

$$\bar{f}(r,z,s) = L\left\{\bar{f}(r,z,t)\right\} = \int_0^\infty f(r,z,t)\, e^{-st} dt \tag{18}$$

$$\tilde{f}(\xi,z,s) = H\left\{\bar{f}(x,z,s)\right\} = \int_0^\infty r \bar{f}(x,z,s)\, J_n(\xi r)\, dr \tag{19}$$

With the aid of (16)-(19), equations (11)-(14) becomes

$$\widetilde{u_r''} = a_{11}\tilde{u}_r + a_{14}\tilde{\psi} + a_{15}\tilde{T} + b_{12}\widetilde{u_z'} + b_{13}\widetilde{\phi_\theta'} \tag{20}$$

$$\widetilde{u_z''} = a_{22}\tilde{u}_z + a_{23}\tilde{\phi}_\theta + b_{21}\widetilde{u_r'} + b_{24}\widetilde{\psi}' + b_{25}\widetilde{\mathrm{T}}' \tag{21}$$

$$\widetilde{\phi_\theta''} = a_{32}\tilde{u}_z + a_{33}\tilde{\phi}_\theta + b_{31}\widetilde{u_r'} \tag{22}$$

$$\widetilde{\psi''} = a_{41}\tilde{u}_r + a_{44}\tilde{\psi} + a_{45}\tilde{T} + b_{42}\widetilde{u_z'} \tag{23}$$

$$\widetilde{\mathrm{T}''} = a_{51}\tilde{u}_r + a_{54}\tilde{\psi} + a_{55}\tilde{T} + b_{52}\widetilde{u_z'} \tag{24}$$

where

$$a_{11} = \left(\frac{\xi^2 + s^2 + s_1\xi^2 s^2}{\delta^2 + s_1 s^2}\right), \quad a_{14} = \frac{p_0\xi}{\delta^2 + s_1 s^2}, \quad a_{15} = -\frac{\xi}{\delta^2 + s_1 s^2}$$

$$a_{22} = \frac{\left(\xi^2\delta^2 + s^2 + s_1\xi^2 s^2\right)}{1 + s_1 s^2}, \quad a_{23} = \frac{-p\xi}{1 + s_1 s^2}, \; a_{32} = \frac{-\xi\delta^{*2}}{1 + \frac{s_1 s^2}{\delta_1^2}}$$

$$a_{33} = \frac{\left(\xi^2 + \frac{s^2}{\delta_1^2} + 2\delta^{*2} + \frac{s_1\xi^2 s^2}{\delta_1^2}\right)}{1 + \frac{s_1 s^2}{\delta_1^2}}, \; a_{41} = \frac{p_0\delta_1^*\xi}{1 + \delta_2^* s_1 s^2}$$

$$a_{44} = \frac{\left(\xi^2 + p_1\delta_1^* + \delta_2^* s^2\left(1 + s_1\xi^2\right)\right)}{1 + \delta_2^* s_1 s^2}, \; a_{45} = \frac{-\bar{\nu}\delta_1^*}{1 + \delta_2^* s_1 s^2}$$

$$a_{51} = \frac{\epsilon\xi s^2\left(1 + \tau_q s + \frac{\tau_q^2}{2}s^2\right)}{Z^*\left(1 + \tau_\nu s\right) + \left(1 + \tau_t s\right)s}, \quad a_{54} = \frac{\epsilon\bar{\nu}s^2\left(1 + \tau_q s + \frac{\tau_q^2}{2}s^2\right)}{Z^*\left(1 + \tau_\nu s\right) + \left(1 + \tau_t s\right)s}$$

$$a_{55} = \frac{\left(\xi^2\left(Z^*\left(1 + \tau_\nu s\right) + s\left(1 + \tau_t s\right)\right) + Q^* s^2\left(1 + \tau_q s + \frac{\tau_q^2}{2}s^2\right)\right)}{Z^*\left(1 + \tau_\nu s\right) + \left(1 + \tau_t s\right)s}$$

$$b_{12} = \frac{\xi\left(1 - \delta^2\right)}{\delta^2 + s_1 s^2}, \quad b_{13} = \frac{p}{\delta^2 + s_1 s^2}, \; b_{21} = \frac{-\xi\left(1 - \delta^2\right)}{1 + s_1 s^2}, \; b_{24} = \frac{-p_0}{1 + s_1 s^2}$$

$$b_{25} = \frac{1}{1 + s_1 s^2}, \; b_{31} = \frac{-\delta^{*2}}{1 + \frac{s_1 s^2}{\delta_1^2}}, \; b_{42} = \frac{p_0\delta_1^*}{1 + \delta_2^* s_1 s^2}, \; b_{52} = \frac{\epsilon s^2\left(1 + \tau_q s + \frac{\tau_q^2}{2}s^2\right)}{Z^*\left(1 + \tau_\nu s\right) + \left(1 + \tau_t s\right)s}$$

$$c_2^2 = \frac{\mu + K}{\rho}, \quad \delta^2 = \frac{c_2^2}{c_1^2}, \quad p = \frac{K}{\rho c_1^2}, \quad p_0 = \frac{\lambda_0}{\rho c_1^2}, \; s_1 = \frac{\varepsilon^2\omega^{*2}}{c_1^2}, \; \delta^{*2} = \frac{Kc_1^2}{\gamma\omega^{*2}}, \; \delta_1^2 = \frac{c_3^2}{c_1^2}$$

$$c_3^2=\frac{\gamma}{\rho j},\ \delta_1^*=\frac{\rho c_1^4}{\alpha_0\omega^{*2}},\quad \bar{\nu}=\frac{m}{\nu},\quad p_1=\frac{\lambda_1}{\rho c_1^2},\ \delta_2^*=\frac{\rho j_0 c_1^2}{2\alpha_0},\ Q^*=\frac{\rho C^* c_1^2}{K_1^*\omega^*},\ \epsilon=\frac{\nu^2 T_0}{\rho K_1^*\omega^*}$$

The system of equations (20)-(24) can be written as

$$\frac{d}{dz}W\left(\xi,\ z,\ s\right)=A\left(\xi,\ s\right)W\left(\xi,\ z,\ s\right) \tag{25}$$

where

$$W=\begin{bmatrix} U \\ DU \end{bmatrix},\quad A=\begin{bmatrix} O & I \\ A_2 & A_1 \end{bmatrix},\quad U=\begin{bmatrix} \tilde{u}_r \\ \tilde{u}_z \\ \tilde{\phi}_\theta \\ \tilde{\psi} \\ \tilde{T} \end{bmatrix},\quad D=\frac{d}{dz}$$

$$O=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, I=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2=\begin{bmatrix} a_{11} & 0 & 0 & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 \\ a_{41} & 0 & 0 & a_{44} & a_{45} \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix},\ A_1=\begin{bmatrix} 0 & b_{12} & b_{13} & 0 & 0 \\ b_{21} & 0 & 0 & b_{24} & b_{25} \\ b_{31} & 0 & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 & 0 \\ 0 & b_{52} & 0 & 0 & 0 \end{bmatrix}$$

The solution of (25) be taken

$$W\left(\xi,\ z,\ s\right)=X\left(\xi,\ s\right)e^{qz} \tag{26}$$

which yield

$$A\left(\xi,\ s\right)W\left(\xi,\ z,\ s\right)=qW\left(\xi,\ z,\ s\right)$$

which leads to the eigenvalue problem. The characteristic equation corresponding to the matrix A is given by

$$\det\left(A-qI\right)=0$$

which on expansion gives

$$q^{10}-\lambda_1 q^8+\lambda_2 q^6-\lambda_3 q^4+\lambda_4 q^2-\lambda_5=0 \tag{27}$$

where $\lambda_1,\ \ \lambda_2,\ \ \lambda_3,\ \ \lambda_4$ and λ_5 are given in appendix I.
Let the roots of equation (27) be $\pm q_i,\quad i=1,\ 2,\ 3,\ 4,\ 5.$

The eigenvectors $X_i(\xi,\ s)$ corresponding to the eigenvalues q_i can be determined by solving

$$[A - qI]\, X_i(\xi,\ s) = 0$$

The set of eigenvectors $X_i(\xi,\ s)$ may be written as

$$X_i(\xi,\ s) = \begin{bmatrix} X_{i1}(\xi,\ s) \\ X_{i2}(\xi,\ s) \end{bmatrix}$$

where

$$X_{i1}(\xi,\ s) = \begin{bmatrix} a_i q_i \\ b_i \\ -\xi \\ d_i \\ e_i \end{bmatrix},\ X_{i2}(\xi,\ s) = \begin{bmatrix} a_i q_i^2 \\ b_i q_i \\ -\xi q_i \\ d_i q_i \\ e_i q_i \end{bmatrix},\ q = q_i; i = 1,2,3,4,5$$

$$X_{j1}(\xi,\ s) = \begin{bmatrix} -a_i q_i \\ b_i \\ -\xi \\ d_i \\ e_i \end{bmatrix},\ X_{j2}(\xi,\ s) = \begin{bmatrix} a_i q_i^2 \\ -b_i q_i \\ \xi q_i \\ -d_i q_i \\ -e_i q_i \end{bmatrix},\ j = i+5, \quad q = -q_i; i = 1,2,3,4,5$$

where $a_i,\ b_i,\ d_i,\ e_i,\ \Delta_i,\ r_1,\ r_2,\ r_3,\ r_4$ and r_5 are given in appendix II.
We assume the solution of equation (25) as

$$W(\xi,\ z,\ s) = \sum_{i=1}^{5} N_i X_i(\xi,\ s) \cosh(q_i z) \tag{28}$$

where $N_1,\ N_2,\ N_3,\ N_4$ and N_5 are arbitrary constants.

4. Boundary conditions

A concentrated normal force and thermal source are acting on the surface of the plate $z = \pm d$ along with the vanishing of shear stress, couple shear stress and normal microstretch component. Mathematically, these conditions are defined as

$$\frac{dT}{dz} = \pm g_0 F(r,\ z) \tag{29}$$

where $F(r,\ z) = z^2 e^{-\omega r}, \quad \omega > 0$, $F(r,\ z)$ is a function that increases in the axial direction symmetrically and falls off exponentially as one moves away from the center of the plate along the radial direction. g_0 is the constant.

$$t^c_{zz} = \delta(t)\, \delta(a - r) \tag{30}$$

where $\delta()$ is the Dirac delta function.

$$t^c_{zr} = 0 \tag{31}$$

$$m^c_{z\theta} = 0 \tag{32}$$

$$\lambda^{*c}_z = 0 \tag{33}$$

where $t^c_{zz} = \left(1 - \varepsilon^2\nabla^2\right) t_{zz}$, $t^c_{zr} = \left(1 - \varepsilon^2\nabla^2\right) t_{zr}$, $m^c_{z\theta} = \left(1 - \varepsilon^2\nabla^2\right) m_{z\theta}$, $\lambda^{*c}_z = \left(1 - \varepsilon^2\nabla^2\right) \lambda^*_z$ are given by

$$t^c_{zz} = (\lambda + 2\mu + K)\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) - \nu\left(1 + \tau_1\frac{\partial}{\partial t}\right) T + \lambda_0\psi \tag{34}$$

$$t^c_{zr} = (\mu + K)\frac{\partial u_r}{\partial z} + \mu\frac{\partial u_z}{\partial r} - K\phi_\theta \tag{35}$$

$$m^c_{z\theta} = b_0\frac{\partial \psi}{\partial r} + \gamma\frac{\partial \phi_\theta}{\partial z} \tag{36}$$

$$\lambda^{*c}_z = \alpha_0\frac{\partial \psi}{\partial z} - b\frac{\partial \phi_\theta}{\partial z} \tag{37}$$

The expressions of displacements, microrotation, microstretch, temperature distribution and stresses are obtained in the transformed domain with the aid of (1)-(3), (16)-(19) and (28)-(37) as

$$\left(\tilde{u}_r,\quad \tilde{u}_z,\quad \widetilde{\phi}_\theta,\ \widetilde{\psi},\ \tilde{T}\right) = \frac{1}{\Delta}\sum_{i=1}^{5}\left(a_i q_i,\quad b_i,\ -\xi,\quad d_i,\quad e_i\right)\Delta_i \cosh\left(q_i z\right) \tag{38}$$

$$\left(\widetilde{t_{zz}},\quad \widetilde{t_{zr}},\quad \widetilde{m_{z\theta}}\right) = \frac{1}{\Delta}\sum_{i=1}^{5}\left(L_i,\quad M_i,\quad P_i\right)\Delta_i \cosh\left(q_i z\right) \tag{39}$$

where

$$\Delta = \begin{vmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ T_1 & T_2 & T_3 & T_4 & T_5 \\ U_1 & U_2 & U_3 & U_4 & U_5 \\ V_1 & V_2 & V_3 & V_4 & V_5 \\ W_1 & W_2 & W_3 & W_4 & W_5 \end{vmatrix}$$

and $\Delta_i\ (i = 1, 2, 3, 4, 5)$ are obtained from Δ by replacing i^{th} column of Δ with $|Q,\ R,\ 0,\ 0,\ 0|^{tr}$, also

$$S_i = e_i q_i \sinh\left(q_i d\right), \quad T_i = L_i \cosh\left(q_i d\right), \quad U_i = M_i \cosh\left(q_i d\right), \quad V_i = P_i \cosh\left(q_i d\right)$$

$$W_i = Q_i \sinh(q_i d), \; Q = \pm g_0 \frac{z^2\omega}{(z^2+\omega^2)^{3/2}}, \; R = aJ_0(\xi a)$$

$$L_i = \left(1 + s^2\left(\xi^2 - q_i^2\right)\right)\left(\frac{\lambda \xi a_i q_i}{\rho c_1^2} + p_0 d_i - e_i + b_i q_i\right), \quad i = 1,\ 2,\ 3,\ 4,\ 5$$

$$M_i = \left(-\frac{\mu \xi b_i}{\rho c_1^2} + \frac{\xi K}{\rho c_1^2} + \left(\frac{\mu + K}{\rho c_1^2}\right) a_i q_i^2\right), \qquad i = 1,\ 2,\ 3,\ 4,\ 5$$

$$P_i = \left(\frac{-\gamma \xi \omega^{*2}}{\rho c_1^4}\right)(p_0 d_i + q_i), \quad i = 1,\ 2,\ 3,\ 4,\ 5$$

$$Q_i = \frac{\omega^{*2} q_i}{\rho c_1^4}(\alpha_0 d_i + b_0 \xi), \quad i = 1,\ 2,\ 3,\ 4,\ 5$$

5. Particular cases

1. Taking $\varepsilon = 0$, in equation (38)-(39), yield the corresponding results for microstretch thermoelastic medium with three phase lag model.
2. Taking $K^* = 0$, in equation (38)-(39), yield the corresponding results for nonlocal microstretch thermoelastic medium with dual phase lag model.
3. Taking $K^* = 0,\ \tau_t = \tau_q^2 = 0$ and $\tau_q = \tau_0$, in equation (38)-(39), yield the corresponding results for nonlocal microstretch thermoelastic with one relaxation time.
4. Taking $\tau_\upsilon = K_1^* = \tau_q = \tau_q^2 = 0$, in equation (38)-(39), yield the corresponding results for nonlocal microstretch thermoelastic medium with energy dissipation.
5. Taking $\tau_\upsilon = \tau_t = \tau_q = \tau_q^2 = 0$, in equation (38)-(39), yield the corresponding results for nonlocal microstretch thermoelastic medium without energy dissipation.

6. Inversion of transforms

We have to obtain the transformed displacements, microrotation, microstretch, temperature distribution and stresses in the physical domain, so, we invert the transforms in the resulting expressions (38)-(39). All these expressions are functions of the form $\tilde{f}(\xi, z, s)$. Therefore, we get the function $f(r, z, t)$ by using the inversion of the Hankel and Laplace transforms are defined by

$$\tilde{f}(\xi, z, s) = \int_0^\infty \xi \bar{f}(\xi, z, s) J_n(\xi r)\, d\xi \tag{40}$$

$$f(r, z, t) = \frac{1}{2\iota\pi} \int_{c-\infty}^{c+\infty} \bar{f}(r, z, s)\, e^{-st} ds \tag{41}$$

where c is an arbitrary constant greater than all real parts of the singularities of $\bar{f}(r, z, t)$.

7. Numerical results and discussions

Following Kiris and Inan (2008), Tomar and Khurana (2008), the values of parameters for nonlocal microstretch elastic solid are given by

$$\lambda = 7.59 \times 10^9 Nm^{-2}, \quad \mu = 41.90 \times 10^9 Nm^{-2}, \quad K = 1.3234 \times 10^5 Nm^{-2},$$

$$\rho = 2192 kgm^{-3}, \quad j, j_0 = 0.196 \times 10^{-6} m^2, \quad e_0 = 0.39, \quad a = 0.5 \times 10^{-9} m,$$

$$\alpha = 8.3255 \times 10N, \quad \beta = 0.10282 \times 10^3 N, \quad \gamma = 0.779 \times 10^{-9} N,$$

$$\alpha_0 = 15.947 \times 10^3 N, \quad b_0 = 0.096 \times 10^6 N, \quad \lambda_0 = 0.57702 \times 10^3 N,$$

$$\lambda_1 = 34.650 \times 10^3 N$$

Following Dhaliwal and Singh (1980) give the values for thermal parameters as

$$C^* = 1.04 \times 10^3 JKg^{-1}K^{-1}, \quad K_1^* = 1.7 \times 10^6 Jm^{-1}s^{-1}K^{-1}, \quad \alpha_t = 2.33 \times 10^{-5} K^{-1},$$

$$\tau_t = 0.1s \times 10^{-13} sec, \quad \tau_q = 0.2s \times 10^{-13} sec,$$

$$\tau_0 = 6.131 \times 10^{-13} sec, \quad \tau_1 = 8.765 \times 10^{-13} sec,$$

$$T_0 = 0.298 \times 10^3 K, \quad m = 1.13849 \times 10^{10} N/m^2 K, \quad t = 0.01 sec$$

Figures 1-5 represent the variations of normal stress, couple stress, couple shear stress, microstretch and temperature distribution with distance r in case of nonlocal microstretch thermoelastic with three phase lag model (NMT), microstretch thermoelastic with three phase lag model (NM), nonlocal microstretch thermoelastic with dual phase lag model (NMD), nonlocal microstretch thermoelastic with Lord Shulman theory (NML), nonlocal microstretch coupled thermoelasticity (NCT), microstretch coupled thermoelasticity (WNCT), nonlocal microelongated thermoelastic (NMT) and microelongated thermoelastic (WNMT). In all these figures, NLS, WNLS, NGL, WNGL, NCT, WNCT, NMT and WNMT corresponding to solid line (——), solid line with centred symbol (—*—*—), dash line(—–), dash line with centred symbol (-*-*-*-*-), dash line (— - — -), dash line with centred symbol (— -*— -*), dash line (— — —) and dash line with centred symbol (—*—*—) respectively.

Figure 1 exhibits that the values of t_{zz} initially increase sharply for $1 \leq r \leq 2.2$ and then oscillate for $2.2 \leq r \leq 6$ for NMT, NMD and NMIII. The values of t_{zz} increase for $1 \leq r \leq 1.2$, decrease for $1.2 \leq r \leq 2.2$ and then oscillate for $2.2 \leq r \leq 6$ for MT and NMII. Its value decreases for $1 \leq r \leq 2.5$, increases for $2.5 \leq r \leq 5$ and then becomes stationary for $5 \leq r \leq 6$. It has been seen that the value of t_{zz} for MT is almost same as NMII except in the range $1.2 \leq r \leq 3$. It is clear from the figure that the magnitude of t_{zz} for NMT is lesser than that of all other cases at the beginning.

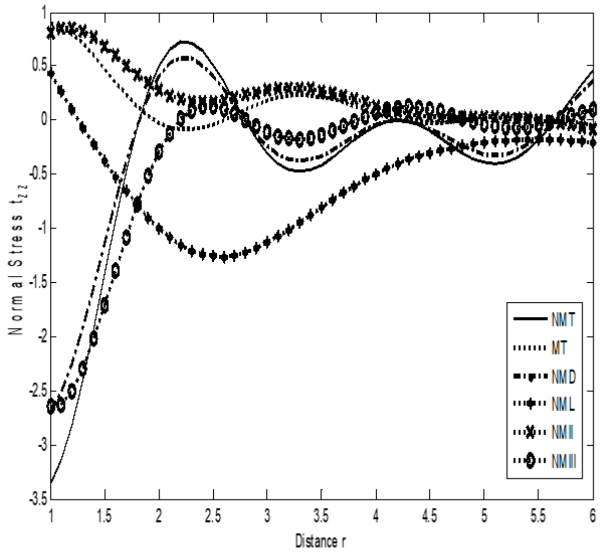

Fig. 1 – Variations of normal stress t_{zz}.

Figure 2 displays that the values of t_{zr} initially decrease for $1 \leq r \leq 1.6$ and then oscillate for the remaining range for NMT, NMD and NMIII. t_{zr} increase for $1 \leq r \leq 1.6$, $1 \leq r \leq 1.8$ for NM and NMII and then oscillates. The oscillatory behavior is also noticed in the case of NML. The values are similar for NMT and NMIII except for $1.2 \leq r \leq 2$. Also it can be noticed that the values of t_{zr} are maximum in the case of NM and minimum in the case of NMIII. It is observed that the variation of t_{zr} is oscillatory in nature, but the amplitude of the oscillation is different for each curve.

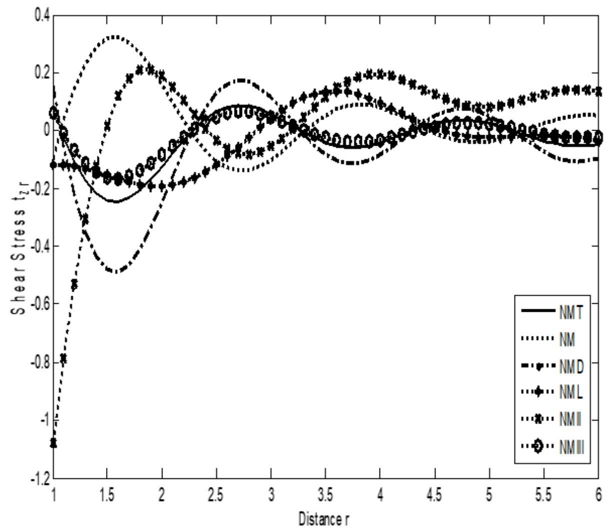

Fig. 2 – Variation of shear stress t_{zr}.

Figure 3 shows that the values of $m_{z\theta}$ sharply decrease for NML for $1 \leq r \leq 1.7$ in comparison to NMT, NMD and NMII and oscillate for these cases for $1.7 \leq r \leq 6$. Its values increase for MT and NMIII and then oscillate with large amplitude. The values of couple shear stress are higher in the cases of NML and NMIII near the application of the source. A similar trend of variation is noticed for NMT, NMD, NML and NMII. It is noticed from the figure that the variation is similar for MT and NMIII.

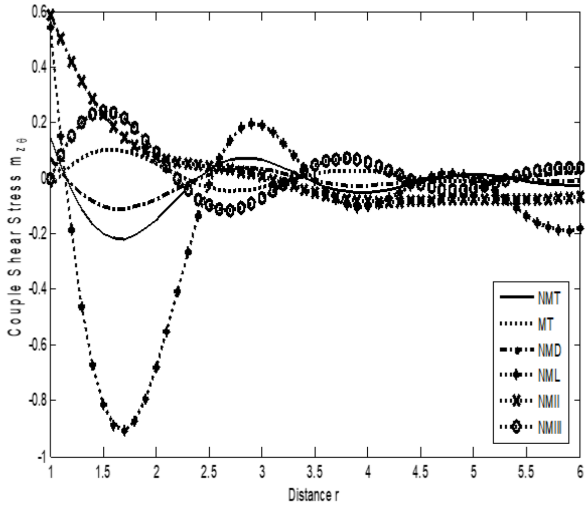

Fig. 3 – Variation of couple shear stress $m_{z\theta}$

Figure 4 exhibits that the values of microstretch ψ first decrease for $1 \leq r \leq 2$, increase rapidly for $2 \leq r \leq 4.7$ and then again decrease smoothly for $4.7 \leq r \leq 6$ for NMT and NMT. Also, its value for NML, initially increase for $1 \leq r \leq 2.7$, decrease for $2.7 \leq r \leq 5.2$ and then its value are stationary for $5.2 \leq r \leq 6$ as the radial distance increases further. Moreover, similar behavior is noticed for MT and NMIII. The behavior of ψ for NMII is opposite to behavior of MT and NMIII. Initially the value for NMIII is large and for NMII is small in comparison to other cases.

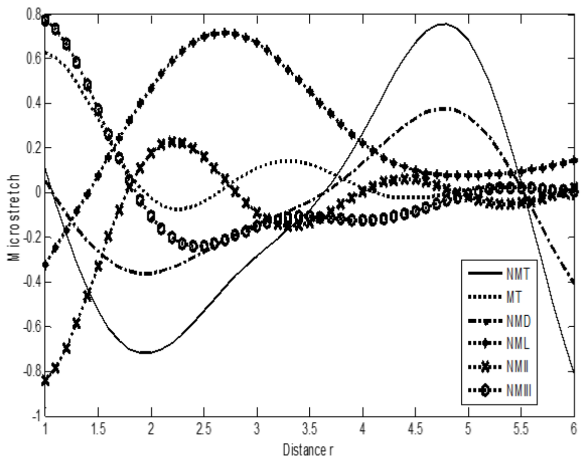

Fig. 4 – Variation of microstretch ψ.

Figure 5 shows that the value of T initially increasing and then decreasing for NMII. Its value initially increases for $1 \leq r \leq 1.2$, decreases for $1.2 \leq r \leq 2.5$ and then oscillates for $2.5 \leq r \leq 6$ for NMT. The values of T initially decrease for $1 \leq r \leq 2.3$ and then oscillate for $2.3 \leq r \leq 6$ for MT, NMD, NML and NMIII. Away from the source, all the quantities have similar variation except for NMII. Near the application of the source, T has similar variation for MT and NML. Initially, the values are large for NMT and small for NMII in comparison to the other cases.

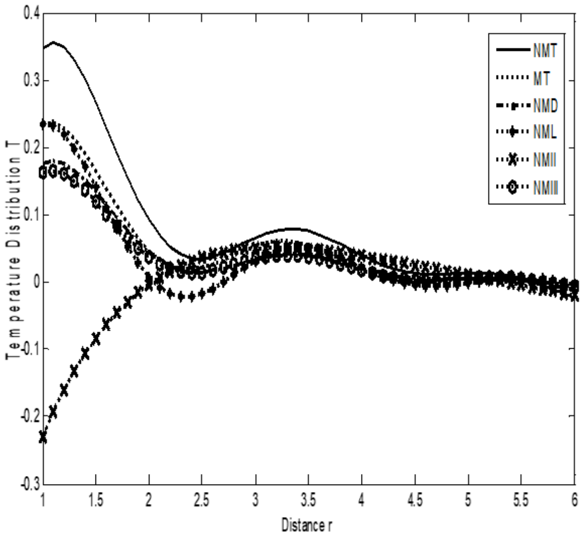

Fig. 5 – Variations of temperature distribution T.

8. Conclusions

In this paper, an axisymmetric problem of microstretch thermoelastic circular plate with three phase lag model by employing eigen value approach subjected to thermomechanical sources has been investigated. Laplace and Hankel transforms are applied to solve the problem. All the resulting quantities are influence by the effect of nonlocal, phase lag, relaxation time, with and without energy dissipation. All the resulting quantities have significant oscillatory behavior. It is observed that the behavior of normal stress and shear stress are similar for NMT, NMD and NMIII while reverse behavior is observed for MT and NMII. However, couple shear stress has also similar behavior for NMT, NMD and NML which is opposite to MT and NMIII. Temperature distribution has similar behavior for all the resulting quantities except for NMII. It is clear from the figure that the values for all the cases are oscillate with different amplitude for microstretch. Its value initially increased and then decreased smoothly for NMII. All the quantities have similar behavior for NMT and NMD. This study is very important for the researchers who work in the field of geophysics, earthquake engineering and for seismologists working in the field of mining tremors and drilling into the earth's crust.

9. Conclusion

In this paper, an axisymmetric problem of microstretch thermoelastic circular plate with three phase lag model by employing eigen value approach subjected to thermomechanical sources has been investigated. Laplace and Hankel transforms are applied to solve the problem. All the resulting quantities are influence by the effect of nonlocal, phase lag, relaxation time, with and without energy dissipation. All the resulting quantities have

significant oscillatory behaviour. It is observed that the behaviour of normal stress and shear stress are similar for NMT, NMD and NMIII while reverse behaviour is observed for MT and NMII. However, couple shear stress has also similar behaviour for NMT, NMD and NML which is opposite to MT and NMIII. Temperature distribution has similar behaviour for all the resulting quantities except for NMII. It is clear from the figure that the values for all the cases are oscillate with different amplitude for microstretch. Its value initially increased and then decreased smoothly for NMII. All the quantities have similar behaviour for NMT and NMD. This study is very important for the researchers who work in the field of geophysics, earthquake engineering and for seismologists working in the field of mining tremors and drilling into the earth's crust.

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Appendix I

$$\lambda_1 = -\left(a_{11} + a_{22} + a_{33} + a_{44} + a_{55} + b_{12}b_{21} + b_{13}b_{31} + b_{25}b_{52} + b_{24}b_{42}\right)$$

$$\begin{aligned}\lambda_2 = {} & -a_{14}a_{41} + a_{33}a_{55} + a_{44}a_{55} + a_{11}a_{55} + a_{22}a_{55} + a_{33}a_{44} + a_{11}a_{33} + a_{22}a_{33} \\ & + a_{11}a_{44} + a_{22}a_{44} + a_{11}a_{22} - a_{15}a_{51} - a_{45}a_{54} - a_{23}a_{32} + \left(a_{33} + a_{44} + a_{55}\right) b_{12}b_{21} \\ & - \left(a_{14}b_{42} + a_{15}b_{52} + a_{32}b_{13}\right) b_{21} + \left(a_{11} + a_{33} + a_{55}\right) b_{42}b_{24} + \left(a_{11} + a_{33} + a_{44}\right) b_{25}b_{52} \\ & - \left(a_{41}b_{24} + a_{23}b_{31} + a_{51}b_{25}\right) b_{12} + \left(a_{22} + a_{44} + a_{55} + b_{42}b_{24} + b_{25}b_{52}\right) b_{31}b_{13} \\ & - a_{45}b_{52}b_{24} - a_{54}b_{42}b_{25}\end{aligned}$$

$$\begin{aligned}\lambda_3 = {} & \left(a_{11}a_{22} + a_{22}a_{55}\right)\left(a_{33} + a_{44}\right) - a_{23}a_{32}\left(a_{11} + a_{44} + a_{55}\right) + a_{11}a_{55}\left(a_{22} + a_{33} + a_{44}\right) \\ & + a_{33}a_{44}\left(a_{11} + a_{22} + a_{55}\right) - a_{45}a_{54}\left(a_{11} + a_{22} + a_{33}\right) - a_{14}a_{41}\left(a_{22} + a_{33} + a_{55}\right) \\ & - a_{15}a_{51}\left(a_{22} + a_{33} + a_{44}\right) + b_{42}b_{25}\left(a_{14}a_{51} - a_{11}a_{54} - a_{33}a_{54}\right) \\ & + b_{52}b_{25}\left(-a_{14}a_{41} + a_{11}a_{33} + a_{14}a_{44} + a_{33}a_{44}\right) + b_{52}b_{24}\left(a_{15}a_{41} - a_{11}a_{45} - a_{33}a_{45}\right) \\ & - b_{12}b_{25}\left(a_{33}a_{51} + a_{44}a_{51} - a_{41}a_{54}\right) - b_{12}b_{24}\left(a_{33}a_{41} - a_{45}a_{51} + a_{41}a_{55}\right) \\ & + b_{42}b_{24}\left(-a_{15}a_{51} + a_{11}a_{33} + a_{11}a_{55} + a_{33}a_{55}\right) - a_{32}b_{21}b_{13}\left(a_{44} + a_{55}\right) \\ & + b_{31}b_{13}\left(a_{22}a_{44} + a_{22}a_{55} + a_{44}a_{55} - a_{45}a_{54}\right) + b_{21}b_{42}\left(a_{15}a_{54} - a_{14}a_{33} - a_{14}a_{55}\right) \\ & + b_{21}b_{52}\left(a_{14}a_{45} - a_{15}a_{33} - a_{15}a_{44}\right) + b_{12}b_{21}\left(-a_{45}a_{54} + a_{33}a_{44} + a_{44}a_{55} + a_{33}a_{55}\right) \\ & - b_{12}b_{31}\left(a_{23}a_{44} + a_{23}a_{55}\right) - b_{31}b_{13}\left(a_{54}b_{42}b_{25} - a_{44}b_{52}b_{25} + a_{45}b_{52}b_{24} - a_{55}b_{24}b_{42}\right) \\ & + a_{32}b_{13}\left(a_{51}b_{25} + a_{41}b_{24}\right) + a_{15}a_{41}a_{54} + a_{14}a_{45}a_{51} + \left(a_{14}b_{42} + a_{15}b_{52}\right) a_{23}b_{31}\end{aligned}$$

$$\begin{aligned}\lambda_4 = {} & \left(a_{44}a_{55} - a_{45}a_{54}\right)\left(a_{33}b_{12}b_{21} - a_{23}b_{12}b_{31}\right) + \left(a_{15}a_{54} - a_{14}a_{55}\right) \\ & \left(a_{33}b_{21}b_{42} - a_{23}b_{31}b_{42}\right) + \left(a_{14}a_{45} - a_{15}a_{44}\right)\left(a_{33}b_{21}b_{52} - a_{23}b_{31}b_{52}\right) \\ & + \left(a_{45}a_{54} - a_{44}a_{55}\right)\left(a_{32}b_{21}b_{13} - a_{22}b_{31}b_{13}\right) + a_{33}b_{12}b_{24}\left(a_{45}a_{51} - a_{41}a_{55}\right) \\ & + a_{33}b_{42}b_{24}\left(-a_{15}a_{51} + a_{11}a_{55}\right) + a_{32}b_{13}b_{24}\left(a_{41}a_{55} - a_{45}a_{51}\right) \\ & + \left(a_{41}a_{54} - a_{44}a_{51}\right)\left(a_{33}b_{12}b_{25} - a_{32}b_{13}b_{25}\right) + a_{33}b_{42}b_{25}\left(a_{14}a_{51} - a_{11}a_{54}\right) \\ & + a_{33}b_{52}b_{25}\left(a_{11}a_{44} - a_{14}a_{41}\right) + a_{15}a_{51}\left(a_{23}a_{32} - a_{22}a_{33} - a_{22}a_{44} - a_{33}a_{44}\right) \\ & + a_{14}a_{45}a_{51}\left(a_{22} + a_{33}\right) + a_{15}a_{41}a_{54}\left(a_{22} + a_{33}\right) + a_{45}a_{54}\left(a_{23}a_{32} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33}\right)\end{aligned}$$

$$+a_{14}a_{41}\left(a_{23}a_{32}-a_{22}a_{33}-a_{22}a_{55}-a_{33}a_{55}\right)-a_{11}a_{23}a_{32}\left(a_{44}+a_{55}\right)$$
$$+a_{55}\left(a_{11}a_{22}a_{33}-a_{23}a_{32}a_{44}\right)+a_{11}a_{22}a_{44}\left(a_{33}+a_{55}\right)+a_{33}a_{44}a_{55}\left(a_{11}+a_{22}\right)$$
$$+\left(a_{15}a_{41}-a_{11}a_{45}\right)a_{33}b_{24}b_{52}$$

$$\begin{aligned}\lambda_5 &= \left(a_{22}a_{33}-a_{23}a_{32}\right)\left(a_{11}a_{44}a_{55}-a_{11}a_{45}a_{54}+a_{14}a_{45}a_{51}\right)\\ &\quad+\left(a_{15}a_{54}-a_{14}a_{55}\right)\left(a_{22}a_{33}a_{41}-a_{23}a_{32}a_{41}\right)+\left(a_{23}a_{32}-a_{22}a_{33}\right)a_{15}a_{44}a_{51}\end{aligned}$$

Appendix I

$$\begin{aligned}a_i &= \frac{\xi}{\Delta_i}[r_1^2\left\{r_2\left(r_3\left(1-\delta^2\right)+p\delta^{*2}\right)-p_0^2\delta_1^*r_3\right\}\\ &\quad+\epsilon s^2r_1r_5\left\{r_3\left(r_2+\bar{\nu}^2\delta_1^*\left(1-\delta^2\right)-2\bar{\nu}p_0\delta_1^*-\bar{\nu}\delta_1^*\right)+p\bar{\nu}^2\delta_1^*\delta^{*2}\right\}]\end{aligned}$$

$$\begin{aligned}b_i &= \frac{-1}{\Delta_i}[r_1^2\left\{\ r_2\left(r_3r_4+p\delta^{*2}q_i^2\right)-p_0^2\delta_1^*\xi^2r_3\right\}\\ &\quad+\epsilon s^2r_3r_1r_5\left(\xi^2\ r_2+\bar{\nu}^2\delta_1^*r_4-2\bar{\nu}p_0\delta_1^*\xi^2\right)+\epsilon s^2p\bar{\nu}^2\delta_1^*\delta^{*2}q_i^2r_1r_5]\end{aligned}$$

$$d_i=\delta_1^*\left(p_0r_1+\epsilon s^2\bar{\nu}r_5\right)\left(\xi a_i+b_i\right)q_i/\left(-r_2r_1-\epsilon s^2\bar{\nu}^2\delta_1^*r_5\right)$$

$$\begin{aligned}e_i &= [\epsilon r_5\{s^2\left(r_2r_1+\epsilon s^2\bar{\nu}^2\delta_1^*r_5\right)\\ &\quad-\bar{\nu}\delta_1^*\left(p_0r_1+\epsilon\bar{\nu}r_5\right)\}]\left(\xi a_i+b_i\right)q_i/\left\{r_1\left(-r_2r_1-\epsilon s^2\bar{\nu}^2\delta_1^*r_5\right)\right\}\end{aligned}$$

$$\begin{aligned}\Delta_i &= \delta^{*2}[r_1^2\left\{r_2\left(\xi^2+s^2-q_i^2\right)+p_0^2\delta_1^*\left(q_i^2-\xi^2\right)\right\}\\ &\quad+\epsilon s^2r_2r_1r_5\left(\xi^2-q_i^2\right)+\epsilon s^2\delta_1^*r_1r_5\left\{\bar{\nu}^2\left(\xi^2+s^2-q_i^2\right)-2p_0\bar{\nu}\left(\xi^2-q_i^2\right)\right\}]\end{aligned}$$

$$\begin{aligned}r_1 &= \left(\left(\xi^2\left(Z^*\left(1+\tau_\nu s\right)+s\left(1+\tau_t s\right)\right)+Q^*s^2\left(1+\tau_q s+\frac{\tau_q^2}{2}s^2\right)\right)\right.\\ &\quad\left.-q_i^2\left(Z^*\left(1+\tau_\nu s\right)+\left(1+\tau_t s\right)s\right)\right)\end{aligned}$$

$$r_2=\left(\xi^2+\delta_3^*s^2+p_1\delta_1^*+\delta_2^*s-q_i^2\right),\ r_3=\left(\xi^2+\frac{s^2}{\delta_1^2}+2\delta^{*2}-q_i^2\right)$$

$$r_4=\left(\xi^2+s^2-\delta^2q_i^2\right),\quad r_5=\left(1+\tau_q s+\frac{\tau_q^2}{2}s^2\right)$$