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Numerical treatment of a nonlinear hyperbolic equation

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ABSTRACT

In this work we consider a nonlinear elliptic partial differential equation, which is derived from an application of a nonlinear Schrödinger equation. Using a variational approach on this problem leads to an optimization problem with a nonlinear constraint. A numerical solution based on finite-element method is used. We propose a new iterative algorithm to relax this problem to a quadratic version.

1. Introduction

The nonlinear Schrödinger equation [1] appears in many physical contexts and applied mathematics [8]. It is a very useful mathematical tool in describing many physical systems in various fieldssuch as the propagation of classical waves in dispersive nonlinear media [9].

It is a partial differential equation which describes the evolution over time of a quantum object represented by a wave function ψ of a given physical system [1, 6, 7]. Also, it takes different forms depending on the interaction potential or according to the type of nonlinearity. However, the nonlinear Schrödinger equation is usually described in dimension d as :

$$i\frac{\partial\psi\left(x,t\right)}{\partial t} = -\frac{1}{2}\Delta\psi\left(x,t\right) + \mu|\psi\left(x,t\right)|^2\psi\left(x,t\right) \tag{1}$$

Wheret is the time, x is a spatial coordinate, the wave function ψ of a system is a postulate, which contains all the information, and unfolds several meanings. The letter i denotes the complex number.

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And we have $x = \sum_{j=1}^{d} r_j e_j$ and $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial^2 r_j}$ is the Laplace operator. The parameter μ characterizes the nonlinearity of the problem[1].

To solve the equation (1), we look for a wave solution [6, 7] which is written in this following form

$$\psi(x,t) = e^{-i\lambda t} u(x) \tag{2}$$

The parameter λ is interpreted physically as the chemical potential of the given system.

By replacing (2) in (1), we get the following Euler-Lagrange equation :

$$-\frac{1}{2}\Delta u + \mu |u|^2 u = \lambda u \tag{3}$$

In the case, where G is equal to the positive constant μ, we will obtain the following problem with constant that we will study in this paper.

We consider “the operator” H of Schrödinger [6] which for all $u \in H_0^1(\Omega)$:

$$\Delta u = -\Delta u + \mu u^3$$

Where Ω is an open bounded subset of $\mathbb{R}^N$ with regular border $\partial\Omega$ and $\mu \in \mathbb{R}^+$. Let the problem be to find a function $u \in H_0^1(\Omega)$ and a real $\lambda \in \mathbb{R}$ solution of :

$$\begin{cases} -\Delta u + \mu u^3 = \lambda u \text{ in } \Omega, \forall \mu \in \mathbb{R}^+ \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{4}$$

This problem can be treated as an eigenvalue problem of the nonlinear "operator" H with homogeneous Dirichlet condition. By using the Green's formula, we obtain the following energy minimization problem

$$\min_{\nu \in K} J(\nu)$$

with $J(\nu) = \frac{1}{2}\int_{\Omega} |\nabla \nu|^2 + \frac{\mu}{4}\int_{\Omega} \nu^4$ for a give $\mu \in \mathbb{R}^+$, and $\nu \in K = \left\{ \nu \in H_0^1(\Omega) / \int_{\Omega} \nu^2 = 1 \right\}$.

Proposition 1. We suppose that $\mu \geq 0$ and if N is such that the injection of $H_0^1(\Omega)$ in $L^4(\Omega)$ is compact, then the functional J is infinite in $H_0^1(\Omega)$, the set K is non empty and weakly closed in $H_0^1(\Omega)$ and J has at least a global minimum in K.

Remark 1. The problem (4) has more then a solution, since, u is a solution, then $-u$ and $|u|$ are also solutions. To construct the iterative algorithm, we suppose that u_k is known at the iteration k and we find u_{k+1}. This is what we examine in the remaining sections of this paper. For this, we posed $g_k = u_k^2$ and we consider the next problem.

$$\begin{cases} -\Delta u_{k+1} + \mu g_k u_{k+1} = \lambda u_{k+1} \text{ in } \Omega, \forall \in \mathbb{R}^+ \\ u_{k+1} = 0 \text{ on } \partial\Omega \end{cases} \tag{5}$$

This problem leads to a nonlinear eigenvalue problem according to the operator $-\Delta +$

$\mu g_k Id$.

Proposition 2. The solution u of the problem (4) is an eigenfunction associated to the smallest eigenvalue of the operator $-\Delta + \mu u^2 Id$. Then, it is solution of

$$\min_{\nu \in K} \frac{1}{2} \int_{\Omega} |\nabla \nu|^2 + \frac{\mu}{2} \int_{\Omega} u^2 \nu^2 \tag{6}$$

Proposition 3. All global minimum of J on K keeps a constant sign and the equation (3) has a unique couple (u, λ) solution such as u is nonnegative.

We have proved that the direct minimization problem (4) with a nonlinear constraint has at most one nonnegative solution, which is also the positive single solution of the equation(3). However, this last equation may admit an infinite solution and change sign.

2. Finite-element approximation

We propose an approach for an optimization of finite element analysis [4, 5]. Therefore, we consider the first order finite-element method given by

$$V_h = \left\{ \nu_h \in C^0([a,b]);\ \nu_{h|[x_i, x_{i+1}]} \in P_1 \text{ and } \nu_h(a) = \nu_h(b) = 0 \right\}$$

Where $i \in \{1, ..., n\}$ and with meshsize $h = \frac{b-a}{n+1}$. Then $V_h \subset H_0^1(\Omega)$, and $\forall \nu_h \in V_h$.

We have :

$$\nu_h(x) \sum_{i=1}^{n} \nu(x_i) \varphi(x_i)$$

with φ_i are the piecewise linear functions.

This method is used to evaluate the objective function with a nonlinear equality constraint in order to find an approximate solution. The problem is discretized as follows :

$$\min_{\nu_h \in K_h} J(\nu_h) = \min_{\nu_h \in K_h} (C_h \nu_h, \upsilon_h) \tag{7}$$

with $C_h = \frac{1}{2}\left(A_h + \frac{\mu}{2} M_{\nu_h}\right)$, $K_h = \left\{ \nu_h \in \mathbb{R}^n;\ (B_h \nu_h, \nu_h) = \int_{\Omega} \nu_h^2 = 1 \right\}$,

$$A_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \quad B_h = \frac{h}{6} \begin{pmatrix} 4 & 1 & \cdots & 0 \\ 1 & 4 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 1 & 4 \end{pmatrix}$$

and M_{ν_h} is a ν dependent symmetric matrix.

Proposition 4. The function $J_h : K_h \to \mathbb{R}^N$; $\nu_h \to J(\nu_h)$ is continuous on $\mathbb{R}^N$ and K_h is a non-empty closed set of $\mathbb{R}^N$, then J_h has at least a minimum in K_h.

3. Iterative method

If u is a solution of (4) associated to the eigenvalue 2λ, or if (u, λ) is a solution of (3) with u is nonnegative, then u is a solution of the following quadratic minimization problem

$$\min_{\nu\in K_u} \frac{1}{2}\int_\Omega |\nabla \nu|^2 + \frac{\mu}{2}\int_\Omega u^2\nu^2$$

with the set $K_u = \left\{ \nu \in H_0^1(\Omega) / \int_\Omega uv = 1 \right\}$. We will hold this note in what follows.

Hence the objective idea of numerical solution of our problem, using the iterative method is to calculate the positive solution of (4) thanks to the following proposed algorithm :

– Choose $v_0 > 0/ \|v_0\|^2_{L^2(\Omega)} = 1$
– For $k \geq 0$ calculate $\tilde{v}_{k+1}$ solution of

$$\min_{\nu\in K_{v_k}} \frac{1}{2}\int_\Omega |\nabla \nu|^2 + \frac{\mu}{2}\int_\Omega v_k^2\nu^2 \tag{8}$$

– $v_{k+1} := \frac{v_{k+1}}{\|\tilde{v}_{k+1}\|_{L^2(\Omega)}}$
– $k = k + 1$
– if $\|v_{k+1} - v_k\|_{L^2(\Omega)} \leq \varepsilon$ or $k \geq k_{\max}$ RETURN
With k_{max} is the maximum number of iterations and ε is a given tolerance.

Proposition 5. Suppose that $\mu = 0$. If $\int_\Omega uv_0 \neq 0$, then the sequence $(v_k)_{k\geq 0}$ converges in $H_0^1(\Omega)$ to $\bar{v} = u$ and it is a single non-negative solution of problem (4).

Proposition 6. For $\mu > 0$, if the sequence $v_k \to \bar{v}$ in $H_0^1(\Omega)$, then $\bar{v}$ is a single nonnegative solution of problem (4).

4. Numerical results

In this section, we proposed some numerical examples. We discuss the performance of our proposed iterative algorithm. That is, while we have made a comparison between the numerical results of this algorithm and the numerical results obtained by the pre-defined function fminconMatlab. The function fmincon serves to minimize a quadratic subject to nonlinear constraints. For $\Omega =]0, 1[$, the function fmincon solves the problem (7) without any difficulties. However, if the number of discretization nodes n exceeds 80, the solution begin to diverge as is shown in figure 1 (left). The figure 1 (right) shows that the proposed method computes the solution and has encountered no problems for $n = 100$. We have chosen $k_{max} = 500$ and $\varepsilon = 10^{-4}$.

For $\mu = 0$, we have a solution associated to the linear calculated operator $-\Delta$. The exact solutioncorresponds to the function $x \to \sqrt{2}\sin(\pi x)$ as a proper function of the linear operator $-\Delta$ associated with the smallest eigenvalue $\lambda = \pi^2 = 9,86960$.

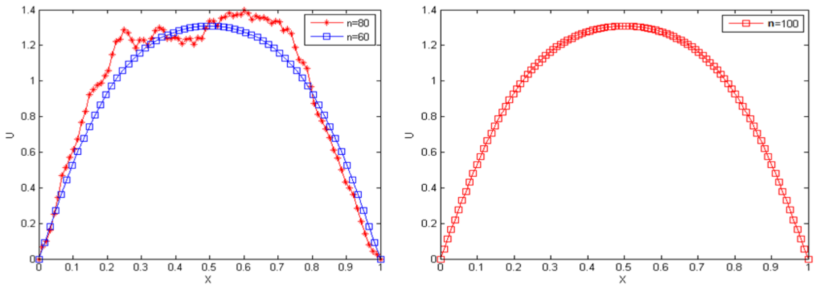

Fig. 1 – Numerical results obtained by both function fminconMatlab (left) and the iterative algorithm for $\mu = 10$ (right).

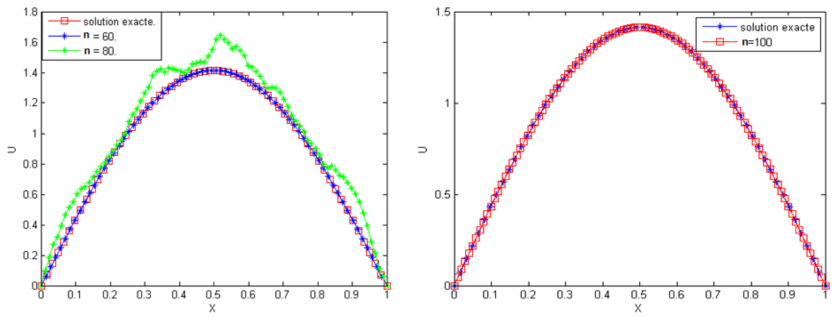

Fig. 2 – Numerical results obtained by both function fminconMatlab (left) and the iterative algorithm for $\mu = 0$ (right).

In figure 3 (left) we can see that the iterative algorithm computes the nonlinear solution with success for $n = 500$ and $\mu = 10$. In addition, the number of iterations used is 124. We can say that the iterative method is more effective in CPU time figure 3 (right).

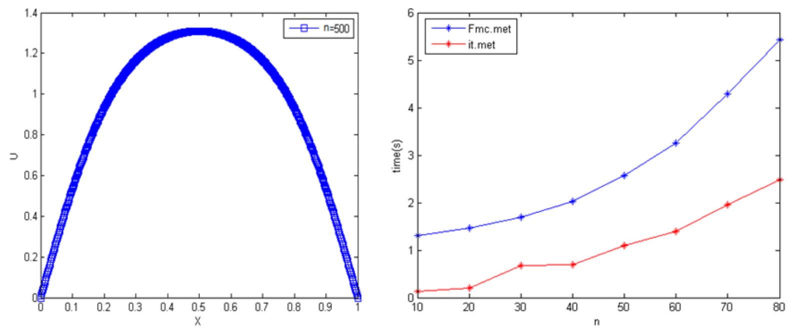

Fig. 3 – Numerical results obtained by iterative algorithm to $n = 500$ and $\mu = 10$ (left). The CPU times presented in blue curve and red curve, respectively, by fmincon function and the proposed method (right).

Without imposing the constraint $v \geq 0$, there is a risk obtain the negative solution which is the opposite of the solution we are looking for. But, the fmincon function give a changing sign solution. This is the case of the following example, where we took $\mu = 10$, $\Omega =]0, 100[$, without considering the constraint $v \geq 0$, we get the solution illustrated in figure 4.

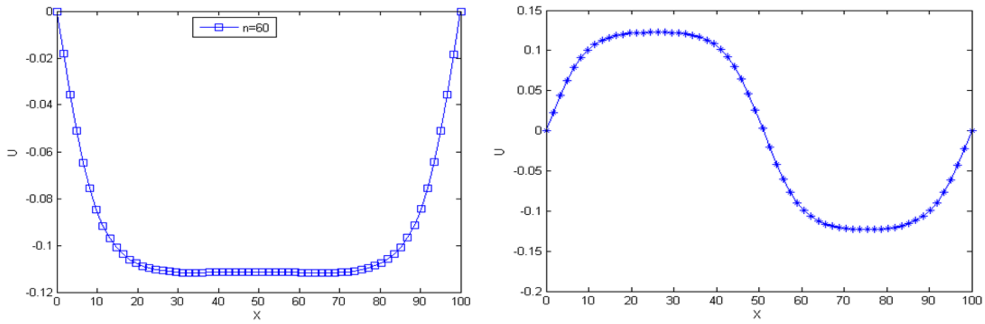

Fig. 4 – Negative and changing signsolutionsfor $n = 60$.

In order to test the effect of the parameter of the problem, we have computed the solution with different values of the size of Ω and μ . We note that when the value μ is very high or when Ω is a large domain the function fminconMatlab and the iterative method give a bad results. If the domain is wide (for example $\Omega =]0, 1000[$) , the number of discretization ($n < 1000$) is insufficient to approximate numerically the sought solution. If the value μ is very important (for example $\mu = 1000$), the term $\int_\Omega |\nabla v|^2$ becomes negligible compared to the term $\mu \int_\Omega v^4$. This explains the divergence of the solution.

The iterative algorithm is more efficient and more optimal than the direct method.

Table 1 – Comparison of the numerical results.

	n	fminconMatlab	Iterative method
$\mu = 0$	60	$\lambda = 9,87$	$\lambda = 9,898$
$\mu = 10$	100	$\lambda = 24,167$	$\lambda = 24,167$

5. Conclusion

In this paper, an eigenvalue problem of a nonlinear hyperbolic operator is studied. We have proved the existence and the uniqueness of a nonnegative solution. In addition, we have proposed a simple iterative to resolve this problem. The numerical result shows its performance in CPU time and in convergence.

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Appendix

Proof of proposition 5

If $\mu = 0$: Let the following minimization problem

$$\min_{v\in K_{v_k}} F(v) = \min_{v\in k_{v_k}} \frac{1}{2}\int_{\Omega} |\nabla v|^2$$

To study the convergence, we go through various steps.

Beginning to prove that the sequence $(\|v_k\|)_{k\geq 0}$ is convergent. In fact, the following inequality is immediate as $\tilde{v}_{k+1}$.

is a minimum of F and v_k is admissible : $F(\tilde{v}_{k+1}) \leq F(v_k)$, which gives $\|\tilde{v}_{k+1}\|^2_{H^1_0(\Omega)} \leq \|v_k\|^2_{H^1_0(\Omega)}$.

Knowing that $v_{k+1} = \frac{\tilde{v}_{k+1}}{\|\tilde{v}_{k+1}\|_{L^2(\Omega)}}$ and using the of Cauchy-Schwartz inequality, we have $0 < \|v_{k+1}\|_{H^1_0(\Omega)} \leq \|v_k\|_{H^1_0(\Omega)}$.

So, the sequence $\left(\|v_k\|_{H^1_0(\Omega)}\right)_{k\geq 0}$ is a decreasing sequence and it is bounded below by 0, therefore it is convergent.

In second step, we show that $\|\tilde{v}_{k+1}\|_{L^2(\Omega)} \underset{k\to\infty}{\to} 1$.

In fact, we know that F_k is $\alpha-$ convex, for $\alpha = 1$.

$F\left(\frac{1}{2}\tilde{v}_{k+1} + \frac{1}{2}v_k\right) + \frac{1}{8}\|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \le \frac{1}{2}F(v_k) + \frac{1}{2}F(\tilde{v}_{k+1})$.

As $\tilde{v}_{k+1}$ is a minimum, then we have

$F(\tilde{v}_{k+1}) + \frac{1}{8}\|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \le \frac{1}{2}F(v_k) + \frac{1}{2}F(\tilde{v}_{k+1})$

$\frac{1}{4}\|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \le F(v_k) - F(\tilde{v}_{k+1}) = \frac{1}{2}\left(\|v_k\|^2_{H_0^1(\Omega)} - \|\tilde{v}_{k+1}\|^2_{H_0^1(\Omega)}\right)$

And since $\|v_{k+1}\|_{H_0^1(\Omega)} \le \|\tilde{v}_{k+1}\|_{H_0^1(\Omega)}$, then

$\frac{1}{2}\|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \le \|v_k\|^2_{H_0^1(\Omega)} - \|\tilde{v}_{k+1}\|^2_{H_0^1(\Omega)}$

According to the previous conclusion, we find $\|\tilde{v}_{k+1} - v_k\|_{H_0^1(\Omega)} \underset{k\to\infty}{\to} 0$.

Using the inequality of Poincaré : $\exists C_p > 0$ such as

$\left|\|\tilde{v}_{k+1}\|_{L^2(\Omega)} - \|v_k\|_{L^2(\Omega)}\right| \le \|\tilde{v}_{k+1} - v_k\|_{L^2(\Omega)} \le C_p\|\tilde{v}_{k+1} - v_k\|_{H_0^1(\Omega)}$,

then $\|v_{k+1} - v_k\|_{H_0^1(\Omega)} \underset{k\to\infty}{\to} 0$.

In the third step, the sequence $(\tilde{v}_{k+1})_k$ is bounded in $H_0^1(\Omega)$ and the real sequence $(\lambda_k)_k = \left(\|\tilde{v}_{k+1}\|^2_{H_0^1(\Omega)}\right)$is also bounded.

For a subsequence, $\exists \bar{v} \in H_0^1(\Omega)$ and $\bar{\lambda} \in \mathbb{R}$ such as $v_k \to \bar{v}$ in $H_0^1(\Omega)$ and $\lambda_k \underset{k\to\infty}{\to} \bar{\lambda}$ in $\mathbb{R}$.

Thanks to the compact injection of $H_0^1(\Omega)$ into $L^2(\Omega)$, we have $v_k \underset{k\to\infty}{\to} \bar{v}$ in $L^2(\Omega)$. Since we have

$-\Delta\tilde{v}_{k+1} = \lambda_k v_k$, thus

$$\int_\Omega \nabla\tilde{v}_{k+1}\nabla v = \lambda_k \int_\Omega v_k v,\ \forall v \in D(\Omega)$$

When we tender k to $+\infty$, we obtain

$$\int_\Omega \nabla\bar{v}\nabla v = \bar{\lambda}\int_\Omega \bar{v}v,\ \forall v \in D(\Omega)$$

So $-\Delta\bar{v} = \bar{\lambda}\bar{v}$. Like this $v_k \ge 0$ p. p. and not null, therefore $\bar{v} \ge 0$.

The uniqueness of a positive proper function implies that $\bar{v} = u$ and $\lambda = \bar{\lambda}$.

Finally, the sequence $(v_k)_k$ converges weakly to u in $H_0^1(\Omega)$ and it is as $\|v_k\|_{H_0^1(\Omega)} \underset{k\to\infty}{\to} \|u\|_{H_0^1(\Omega)}$, in [3] we have $(v_k)_k$ converges strongly to u in $H_0^1(\Omega)$.

Proof of proposition 6

If $\mu > 0$: The steps of this proof are similar to the case $\mu = 0$, but assuming here that the sequence $(v_k)_k$ converges strongly to $\bar{v}$ in $H_0^1(\Omega)$.

We recall that $\tilde{v}_{k+1}$ is a solution of problem

$$\min_{\int_\Omega v v_k = 1} F_k(v) = \min_{\int_\Omega v v_k = 1} \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{\mu}{2}\int_\Omega v_k^2 v^2$$

We beginning by the step, to prove that $\|\tilde{v}_{k+1}\|_{L^2(\Omega)} \underset{k\to\infty}{\to} 1$. Such as

$v_{k+1} = \frac{\tilde{v}_{k+1}}{\|\tilde{v}_{k+1}\|_{L^2(\Omega)}}$,

then, using the inequality of Cauchy-Schwartz, we have
$1 = \int\limits_\Omega \tilde{v}_{k+1} v_k \leq \int\limits_\Omega v_k^2 \int\limits_\Omega \tilde{v}_{k+1}^2, \int\limits_\Omega v_k^2 = 1$ i.e.
$\|\tilde{v}_{k+1}\|_{L^2(\Omega)} \geq 1.$
Besides, we have F_k is strongly convex and $\tilde{v}_{k+1}$ achieves a minimum for F_k, then
$\frac{1}{2} F(\tilde{v}_{k+1}) + \frac{1}{8} \|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \leq \frac{1}{2} F(v_k)$
So,
$\frac{1}{8} \|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \leq \frac{1}{2} F(v_k) - \frac{1}{2} F(\tilde{v}_{k+1})$
giving
$\frac{1}{4} \|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \leq F(v_k) - \left(\int\limits_\Omega \tilde{v}_{k+1}^2 \right) F(v_{k+1})$

and since $\int\limits_\Omega \tilde{v}_{k+1}^2 \geq 1$, the
$\frac{1}{4} \|\tilde{v}_{k+1} - v_k\|^2_{H_0^1(\Omega)} \leq F(v_k) - F(v_{k+1})$
$\leq \frac{1}{2} \int\limits_\Omega |\nabla v_k|^2 - \frac{1}{2} \int\limits_\Omega |\nabla v_{k+1}|^2 + \frac{\mu}{2} \int\limits_\Omega v_k^2 \left(v_k^2 - v_{k+1}^2\right)$
$\leq \frac{1}{2} \int\limits_\Omega |\nabla v_k|^2 - \frac{1}{2} \int\limits_\Omega |\nabla v_{k+1}|^2 + \frac{\mu}{2} \int\limits_\Omega v_k^4 \int\limits_\Omega \left(v_k^2 - v_{k+1}^2\right)^2$
Like that $v_{k+1} \underset{k\to\infty}{\to} \bar{v}$ in $H_0^1(\Omega)$ and the injection of $H_0^1(\Omega)$ in $L^4(\Omega)$ is continuous from Rellich- Kondrachov (see [5]), then we have
$\|\tilde{v}_{k+1} - v_k\|_{H_0^1(\Omega)} \underset{k\to\infty}{\to} 0$
Using the inequality of Poincaré : $\exists C_p > 0$ such as
$\left| \|\tilde{v}_{k+1}\|_{L^2(\Omega)} - \|v_{k+1}\|_{L^2(\Omega)} \right| \leq \|\tilde{v}_{k+1} - v_k\|_{L^2(\Omega)} \leq C_p \|\tilde{v}_{k+1} - v_k\|_{H_0^1(\Omega)}$
Hence the result
$\|v_{k+1}\|_{L^2(\Omega)} \underset{k\to\infty}{\to} 1$, such as $\|v_k\|_{L^2(\Omega)} = 1$.
Thus, we have $\tilde{v}_{k+1} = \|\tilde{v}_{k+1}\|_{L^2(\Omega)} \underset{k\to\infty}{\to} \bar{v}$ in $H_0^1(\Omega)$, therefore
$\tilde{v}_{k+1} = \bar{v} \underset{k\to\infty}{\to}$ in $H_0^1(\Omega)$
In the second step, we note that $\|\bar{v}\|_{L^2(\Omega)} = 1 = \lim\limits_{k\to\infty} \|v_k\|_{L^2(\Omega)}$. Let $\bar{\lambda} = \lim\limits_{k\to\infty} \lambda_k$
Let us demonstrated that $\left(\bar{v}, \bar{\lambda}\right)$ is a solution.
In fact,
$-\tilde{v}_{k+1} + \mu v_k^2 \tilde{v}_{k+1} = \lambda_k v_k$
multiplying this equation by a test function $v \in D(\Omega)$ and using the Greens formula, we obtain
$\int\limits_\Omega \nabla \tilde{v}_{k+1} \nabla v + \mu \int\limits_\Omega v_k^2 \tilde{v}_{k+1} v = \lambda_k \int\limits_\Omega v_k v$
We already have, $(v_k)_k$ and $(\tilde{v}_{k+1})_k$ are two sequence which strongly converge to $\bar{v}$ in

$H_0^1(\Omega)$ and $\lambda_k = \int\limits_\Omega |\nabla \tilde{v}_{k+1}|^2 + \mu \int\limits_\Omega v_k^2 \tilde{v}_{k+1}^2$ converge to real $\bar{\lambda}$.

We deduce, by passing to limit, that

$$\int\limits_\Omega \nabla \bar{v} \nabla v + \mu \int\limits_\Omega \bar{v}^3 v = \bar{\lambda} \int\limits_\Omega \bar{v} v.$$

Since the last equality is true for all $v \in D(\Omega)$, then

$$-\Delta \bar{v} + \mu \bar{v}^3 = \bar{\lambda} \bar{v}$$

Thus $\bar{v}$ is a proper function associated with the smallest eigenvalue $\bar{\lambda}$ of the non linear operator $-\Delta + \mu u^2 Id$, since the sequence $(v_k)_k$ is nonnegative, therefore its limit $\bar{v}$ is also nonnegative. Thus $\bar{v} = u$ and $\bar{\lambda} = \lambda$.