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Sensitivity Analysis of Asset Allocation: In The Presence of Correlation

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Abstract
Linearization of portfolio optimization plays a central role in financial studies, since linear problem allows for performing sensitivity analysis. This concept makes it possible to measure the variation of parameters as a result of variation of one parameter in a linear problem, without solving the problem from scratch. Based on the existing literatures, the approach of CVaR (conditional value at risk) method outperforms other methods, therefore in this study CVaR is applied as a constraint to change portfolio optimization problem into a linear problem. The coefficient of objective function of mentioned method for a portfolio includes average of asset returns, which are highly correlated. Here principal component analysis is employed to convert the correlation of the functional relations. An example of stock market is employed to substantiate the validity of method. Finally, we verify that the result of the presented method is closer to the ideal result.

1. Introduction

The portfolio optimization is of great importance in asset management in order to manage investor's exposure to risks. Based on the different risk measurements portfolio optimization problems have been concerned in various studies. The initial model presented by Markowitz (1950) has suffered from the multi- dimension which measures the risk by variance, many attempts had been made to change the portfolio optimization problem into a linear problem. Thereafter, linear portfolio optimization problems have been widely noticed because the related software and solution were effortless. Better yet it will ease the understanding of concept for whom with little mathematical knowledge. Moreover, it will make the sensitivity analysis possible. However, the mean absolute deviation (Konno, Yamazaki, 1991) and the following advanced technique were widely considered but the method presented by Uryasev (2000) was highly populated due to the following advantages. It has been shown that CVaR is a convex risk measurement (Uryasev, 2002). The minimization of CVaR on the other hand leads to a portfolio with a small VaR. Moreover, CVaR could be involved either as a subjective function or as a constraint (Uryasev, 2002).

After proposition of linear portfolio optimization, the question raised that which algorithm has to be applied in order that the answers become more reliable, therefore in 1947 the simplex algorithm by Dantzig presented. The simplex method finds an optimal solution for the LP problem, which has entered the algorithm. After the simplex method, interior point methods (IPMs) were introduced to solve large-scale problems. This method solves problems in polynomial time (Roos *et al*. 1997; Wright, 1997). According on a research in 1984, simplex algorithm surpassed interior point methods (Potra, 2000). Therefore, this method has been widely used in portfolio optimization problems (Liu, 2006; Lim, 2011; Kokoszkiewicz, 1996).

Later, the importance of measuring the perturbation of other parameters as a result of change in one parameter had been considered. The study of questions like this is studied in the area of sensitivity analysis (Murty, 1983; Bazaraa, 2009). The following sensitivity analysis techniques do not consider the correlation among parameters, and includes some disadvantages.

The ordinary sensitivity analysis which was proposed by Koltai and Terlaky (2000) was only solvable by simplex method and also no simultaneous changes of OFC or RHS was allowed. Then the 100% rule was presented which covers the simultaneous change of OFC or RHS (Bradley, 1977) and later Wendell (1982, 1984, 1985) presented the tolerance approach, if this rule is satisfied then the optimal solution remains unchanged. The noteworthy point is that if this rule is not satisfied it is not known for sure whether the current optimal solution changes or not. Then, the symmetric tolerance analysis was presented, which allows for simultaneous and independent RHS or OFC was proposed. It was simple and easy to use. The tolerance is usually small and for medium and large scale problems it is often zero. However, It loses a lot of information on the model (Wendell, 1982, 1984, 1985).Later, Non-symmetric tolerance analysis was presented, this method considered individual percentage change for every RHS parameters or OFC. However, small tolerance and no simultaneous perturbation among parameters are among weak points of this method (Arsham, 1990; Wondolowski, 1991; Wendell, 1992). The Parametric programming method is useful when RHS parameters or OFC depend only on one parameter. The weak point of this study is that no simultaneous perturbation of RHS parameters or OFC is considered (Saaty, 1954, 1955).Thereafter, Multiparametric programming was introduced. In this method RHS parameters or OFC change simultaneously and independently (Ward &Wendell 1990). In addition, sensitivity analysis in portfolio optimization has been concerned in a number of studies (Best, 1991; Koltai, 2011; Arbaiy, 2013).

To eliminate the shortcomings of ordinary sensitivity analysis Hanafizade (2011) represented a noble method that covers correlation among parameters, the method then followed by Shahin 2016. Shahin used Principal component analysis (PCA) to represents correlation among parameters.

PCA method in some researches is applied as a creator of factors (Victor, 2007; Heij, 2008). Besides, the PCA method has known as a useful method in portfolio optimization, as an example, Sakalauskas in 2012 used the forenamed method for portfolio optimization problem. He also stated the superiority of using PCA method in portfolio optimization verses to portfolio optimization without using PCA, because not only a diversified portfolio is provided by using PCA method, the risk of that portfolio decreases because stocks will be chosen from stocks with few risk.

In this paper we aim to perform sensitivity analysis for a linear portfolio optimization problem under CVaR method and simplex algorithm, CVaR method objective function consists of the average returns of the stocks and these parameters are highly correlated. Among the previous studies there is no single study to deal with this problem. We will use the PCA method to tackle the problem and then the sensitivity analyses based on the related formulas are computed.

The paper is organized as follows. In sect. 2 the methodology of sensitivity analysis in the presence of correlation among parameters is introduced. Sec. 3 illustrates the details of sensitivity analysis whit considering correlation among OFC. Sect.4 represents portfolio optimization using CVaR as a constraint. In Sect.5, the sensitivity analysis of real example of portfolio optimization is examined. Finally, the conclusion can be found in Sect.6 and possible future research directions are presented in section 7.

2. Method

In this paper to construct a linear portfolio under CVaR method, first of all the historical data of four stocks (MNST, MAR, FISV, SON.SG) is derived from the related sites and then the logarithmic returns are calculated. Next, the historical returns divided into two parts the latest (newest) data, which is regarded as future data, and the historical data. For solving portfolio optimization with CVaR method the scenarios are needed, we use Mina & Yi Xiao method to generate scenarios, which is historical simulation. The historical monthly prices are used to derive monthly returns. The problem is solved without the latest data.

Next, we use the covariance matrix as the input of a multivariate statistical method called principal component analysis (PCA) in order to convert correlated parameters (OFC) into independent ones, introduced by some functional relations. To apply PCA, the historical returns for each stock in each 3 months classified -season data- and the average of each class is computed because the objective function constructs of average of returns thus the PCA should be applied over average of data. Then, by knowing variations in one of the OFC- we know this variation by comparing the average of historical returns of each stock with the average of its historical returns while the latest data is considered- and using derivatives of the functional relations we can find out the amount of variations in other OFC.

Then the 100% rule should be applied to see whether the basic variables remain unchanged. If the 100% rule is satisfied, we know that the current optimal solution still remains optimal even though all the RHS parameters or OFC have changed. Then to prove the volatility and the power of predicting this method, the result is compared with two other problems, the first one as we call the ideal result which includes the whole data (the historical price includes the latest data). Furthermore, we consider a case in which instead of latest price the average of historical price is implied (except for one stock which takes its real latest data), to that end the historical price of data without the latest data is considered and then average of each stock is calculated then the result of linear portfolio optimization as before is calculated. Finally, these three results are compared. The following flowchart shows the steps of this paper.

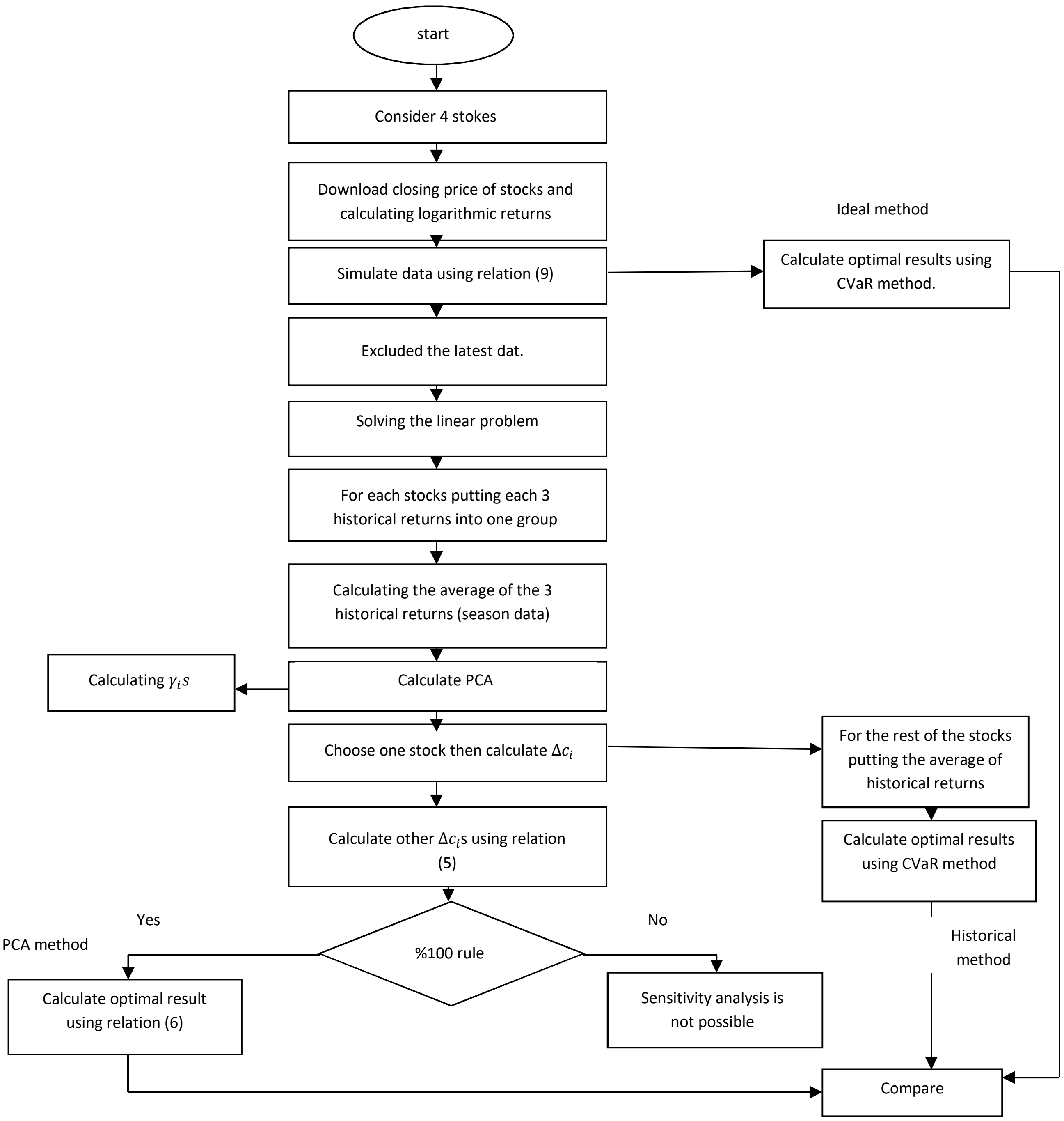

Figure 1. The flowchart of sensitivity analysis in the presence of correlation

3. Sensitivity analysis

In this section we briefly discuss the formulas of sensitivity analysis of LP problem in the presence of correlation among OFC that was introduced in Shahin (2016).

It should be considered that these formulas are relies upon a series of assumptions. First they are valid for local perturbation, that is, the acceptable range for parameters change should be small and within ε-neighborhood of the estimated parameters. Second, it should be noted that these formulas are only applicable when a basic optimal non-degenerate solution is available.

The following LP program is considered as a basis on which other formulas are derived.

$$\text{Min } \boldsymbol{c}^T \boldsymbol{x} \qquad (1)$$
$$\text{s.t. } \mathbf{A}\boldsymbol{x} = \boldsymbol{b},$$
$$\boldsymbol{x} \geq \mathbf{0},$$

Here $\boldsymbol{x}$ is a vector with n variables, **A** is an m*n matrix, $\boldsymbol{c}$ is the OFC vector with n variables, and $\boldsymbol{b}$ is the right hand side (RHS) vector with m parameters.

If program (1) is solved, then the optimal value and the optimal solution calculated as the following:

$$z^* = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b}, \qquad (2)$$
$$\boldsymbol{x}_{\boldsymbol{B}}^* = \boldsymbol{B}^{-1} \boldsymbol{b},$$

Here z^* is the optimal value of the objective function, $\boldsymbol{c}_{\mathbf{B}}$ is the OFC of basic variable and variable $\boldsymbol{x}^*_{\mathbf{B}}$ is the optimal solution. The subscript **B** demonstrates basic variables and the superscript $*$ is indicating optimal value. In addition, **B** is the constraint matrices of basic and N is the constraint matrix of non-basic variables.

If there is not a correlation between the components of vector c, we will have:

$$\frac{dz^*}{dc_k} = \begin{cases} x_{B_i}^*, & if\ c_k = c_{B_i} \\ 0\ , & if\ c_k corresponds\ to\ nonbasic\ variables \end{cases} \qquad (3)$$

In the presence of functional relation between OFC parameters, Shahin (2014) advanced following formulas based on the PCA method in order to change dependent parameters into the independents parameters:

$$\Delta\overline{c}_j = \left[\frac{\gamma_{11}}{\gamma_{1j}} + \frac{\gamma_{21}}{\gamma_{2j}} + \cdots + \frac{\gamma_{n1}}{\gamma_{nj}}\right] \qquad (5)$$

$$\Delta z^* = \left[\frac{\partial z^*}{\partial \overline{c_1}} + \frac{\partial z^*}{\partial \overline{c_2}}\left(\frac{\gamma_{11}}{\gamma_{12}} + \frac{\gamma_{21}}{\gamma_{22}} + \cdots + \frac{\gamma_{n1}}{\gamma_{n2}}\right) + \cdots + \frac{\partial z^*}{\partial \overline{c_n}}\left(\frac{\gamma_{11}}{\gamma_{1n}} + \frac{\gamma_{21}}{\gamma_{2n}} + \cdots + \frac{\gamma_{n1}}{\gamma_{nn}}\right)\right] \Delta \overline{c}_1 \quad (6)$$

Here γ_{ij}s are the entries of the jth eigenvector, which is calculated through PCA method.

4. CVaR Algorithm

In this section we review the CVaR algorithm which is presented by Rockafelar and uryasev (2000). The portfolio optimization under CVaR method is solved by using scenario generation method afterward optimal weights of stocks and optimal value of portfolio could be derived. Then by using the formulas which are presented in the section 3, the portfolio sensitivity analysis based on the small changes in OFC -in CVaR algorithm mean return of stocks price are OFC- will be performed and portfolio sensitivity analysis based on a small changes of each stocks return is determined.

The first step of the CVaR calculation is to find the matrix of historical returns from the matrix of historical prices. We follow Mina and Yi Xiao (2001) historical simulation method, which considers logarithmic returns. Logarithmic returns is the preferred method for return calculations in finance (Eberlein, 2001), and it will make calculations simpler in later stages of the thesis. The general formula for logarithmic returns is as the following:

$$r_{it} = \ln\left(\frac{p_{it}}{p_{it-1}}\right) \quad (8)$$

Here r_{it} is the return of stock i in day t and p_i indicates the initial price of the security, whereas P_{i+1} is the price in the next period.

Then with using formula (9) the scenarios of the next period of stocks price based on the historical monthly returns i.e. r_{it-1}, r_{it-2},... will be calculated.

$$y_{ij} = q_i * exp\left(r_{it-j} * \sqrt{t}\right) \qquad j = 1,2, \dots J \quad (9)$$

Here y_{ij} is the t+1 or the next-month price of stock i in the scenario j also is a random variable and q_i is price of stock i in month t.

The expected end-of-period (t+1) price of stock i is derived from the following equation:

$$E[y_i] = \sum_{j=1}^{J} \pi_j \, y_{ij} = \frac{1}{J}\sum_{j=1}^{J} y_{ij} \quad (10)$$

We assume that all scenarios have equal probability.

Rockafellar and Uryasev (2000) mentioned that CVaR can be considered in an optimization problem as an objective or constraint. If CVaR is considered as the objective of the optimization problem, the risk of the portfolio which measured by CVaR will be minimized based on the given required expected return, and if CVaR is considered as a constraint, the expected return of the portfolio will be maximizing based on the given level of risk.

In this research CVaR will be treated as the constraint of the portfolio optimization, so the objective of the optimization problem is to maximize the portfolio expected return or in other word, we could change the objective function into minimization by adding minus and so we will minimize the portfolio expected loss based on the certain level of risk.

In this case we will have portfolio optimization problem like the following linear program:

$$\min_{X,\zeta} \sum_{i=1}^{n} -E[y_i]x_i \qquad (11)$$

Subject to.

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j e_j \leq w \sum_{i=1}^{n} q_i x_i^0 \qquad (12)$$

$$e_j \geq \sum_{i=1}^{n} \left(-y_{ij} x_i + q_i x_i^0\right) - \zeta. \quad e_j \geq 0. \;\; j = 1, \dots, J \qquad (13)$$

$$q_i x_i \leq v_i \sum_{k=1}^{n} q_k x_k . \quad i = 1, \dots, n \qquad (14)$$

Where i is the number of stocks, $E[y_{ij}]$ is the average of stock's return in all scenarios, $\boldsymbol{x_i}$ is the number of stock i in portfolio, w is the coefficient of risk tolerance, $\boldsymbol{x_i^0}$ is the number of stock i in the initial portfolio, e_j is the coefficient for changing CVaR into the linear variable, q_i is the price of stock i at the end of month in scenario j and ζ is VaR.

5. Numerical Examples

In order to exemplify and mixed two proposed methods, a real example of stock market is solved in this section. Then, the results are compared with the results of two other problems in which correlation among parameters have not been considered.

The data sets are used in this paper are historical monthly close prices of 4 stocks (SON.SG, FISV, MAR, MNST) during February 1, 2014 to September 1, 2015, Then the latest prices are excluded and the problem is solved without the latest data by using CVaR method, for solving portfolio optimization with CVaR method the scenarios are required, thus we use

Mina & Yi Xiao method to generate scenarios, which is historical simulation. First, the historical monthly pries are used to derive monthly returns. 20 scenarios are derived from formula (9) for each stock. We assume that all scenarios have equal likelihood. Then based on the CVaR algorithm the portfolio optimization problem is solved with lingo software. The basic variables are shown in the following table:

Table 1 The basic value

ζ	**e(20)**	**X(MNST)**	**X(FISV)**	**X(MAR)**
692.33	28.53	13.87	22.53	13.59

Because objective function includes mean y, each 3 monthly scenarios is classified, then the average of each class is computed. We regard these data as season data and these data is used to generate PCA.

To do so, first covariance matrix between season data is calculated and then eigenvalue and eigenvector are derived from covariance matrix.

Table 2 Eigenvalues of season data

Component	Eigenvalue
1	0.031548103
2	6.064279805
3	24.62035181
4	1187.936196

It turns out that, the eigenvalue of fourths element is notably bigger than other elements and based on the formula (15) it can be say that 0.97 percent of distribution of data can be explained by the fourth element.

$$s_1^2 + s_2^2 + \cdots + s_p^2 = l_1 + l_2 + \cdots + l_p \tag{15}$$

Where s represents variance and $\boldsymbol{l}$ is the eigenvalue.

Table 3 Eigenvectors of season data

Eigenvectors				
Variable	1	2	3	4
SON1.SG	-0.85846	0.287165	0.380537	0.189148
MNST	0.184078	-0.32384	0.210371	0.903872
FISV	0.311176	0.898522	-0.11826	0.286081
MAR	-0.36378	-0.07294	-0.89272	0.255728

To derive PCA eigenvectors should be ordered based on the eigenvalue, the one which has bigger eigenvalue comes first and so on.

Table 4 PCA of data

γ_{ij}	j=1	j=2	j=3	j=4
i=1	0.189148	-0.38054	0.287165	0.858458
i=2	0.903872	-0.21037	-0.32384	-0.18408
i=3	0.286081	0.118264	0.898522	-0.31118
i=4	0.255728	0.89272	-0.07294	0.363778

Then one stocks is chosen, and for that stock we calculate average y while including latest data (was excluded at first) and the variation of mean y in this case and the initial one (without latest data) is calculated and this is the variation of mean y, it is notable that this variation should be in acceptable range, then the variation of other stock is calculated based on formula 5 and these data should also be in acceptable range. We consider stock 4, the variation between two mean y is -0.242 and variation of other stocks are calculated with formula 5 and the result is shown in table 6:

Table 5 Objective Coefficient Range (from lingo output)

Variable	Allowable Increase	Allowable Decrease
X_SON1.SG	INFINITY	58.96188
X_MNST	37.53784	INFINITY
X_FISV	2.339523	INFINITY
X_MAR	115.8636	2.22285

Table 6 The variation of correlated parameters

Stock (i)	Δc_i
i=1	0.199453733
i=2	1.382213411
i=3	-1.74051599

All of the above changes are in the accepted range for which the basis is unchanged. Now 100% rule should be examined. If it is satisfied, we can calculate the new z*.

$$\sum_{i=1}^{4} \frac{\Delta \bar{c}_j}{\Delta \bar{c}_j^{max}} \leq 1 \quad (16)$$

By replacing the value of $\Delta \bar{c}_j$ in the above equation, it can be seen that the 100% rule is satisfied. Therefore, the current optimal basis remains optimum.

And by calculating $\frac{dz^*}{d\bar{c}_k}$ and replacing them into equations 6 we can calculate the new z^*.

$\Delta z^* = -23.3258$

$new\ z^* = old z^* + \Delta z^* = -4073.013 + (-23.3258) = -4096.3388$

For Comparison we are proceeding two steps. In the first step, we include the latest data that has been excluded first time and the portfolio optimization with whole data is solved. We regard this case as the ideal result because we assume that the future can be predicted 100%. In the second step, which is regarded a historical method the forth stock variation considered as the first case (-0.242) but for the other stocks the average of historical return without the latest data is applied, the intention of this case is that how the result will differ from the ideal result if the correlation among parameters be ignored and the historical data replaced and the derivation of which case is considerable (considering correlation among parameters or using historical data) based on the ideal result. The summary results are presented in table 7.

Table7 The summary of results

	ζ	Non-zero e	x (SON1.SG)	x (MNST)	x (FISV)	x (MAR)	VaR	CVaR	Z*
PCA method	692.33	e(20)=28.53	0	13.87	22.53	13.59	0.16	0.17	-4096.33
Ideal method	468.69	e(21)=127.54	0	14.8018	21.96	13.95	0.11	0.25	-4110.35
Historical method	460.11	105.06	0	15.55135	22.83	11.61	0.111	0.23	-4133.02

As can be identified from the table (7) the variation of PCA method and ideal method is 14.02 and the variation of historical method and the ideal method is 22.26, which can be realized that the PCA method has explained the correlation and has resulted in the closer result to the ideal result. In the following graphs the results of three different cases will be depicted:

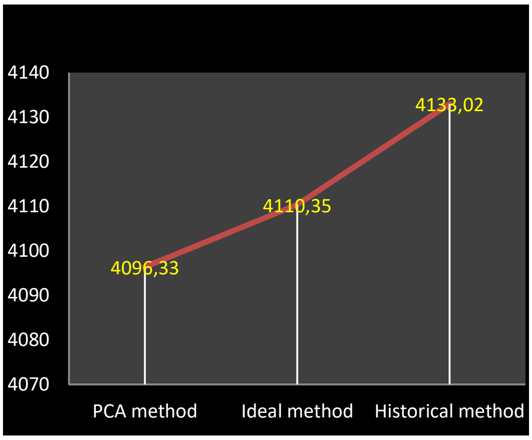

Figure 2. The amount of Z* of three cases

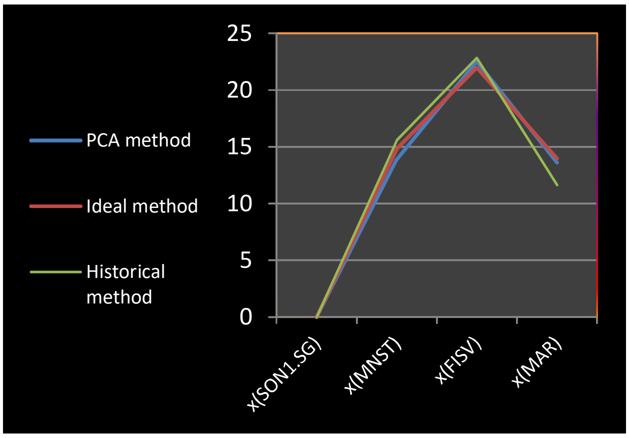

Figure 3. The amount of x*s in optimal portfolio

6. Conclusions

In this study we discussed a linear portfolio optimization regarding correlation among the average returns of four stocks. In case of presence of correlation among prices, the variation of one variable contributes to the variation of correlated variables. Current sensitivity analysis had failed to predict the exact changes in other variables. Here we calculated the changes in correlated variables and compared the result of new method with two other cases.

For performing portfolio optimization the CVaR method is employed and PCA method is used in order to eliminate correlation among OFC. Then, by using related formulas of this method the sensitivity analysis of linear portfolio optimization has performed. For performing sensitivity analysis we divided the historical data into two parts the first part which is called latest data first excluded and the results derived then to illustrate the validity of result we considered two different cases, first one which is called ideal result that is obtained by considering the whole data (including latest data), which we try to be as much as close to this result and second one is the one which obtained through historical data (no correlation is included),in which a stock variation is derived based on the latest data (only one stock) and assumed that other stocks will continue historical behavior, the results then indicate that the result of new sensitivity analysis is closer to the ideal result whereas the historical method.

7. Future Researches

Further research can be done in order to make the result of this study closer to the result of ideal case. To that end, the existent error of correlation matrix should be eliminated by applying random matrix theory.

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