Congruence Properties of Andrews' SPT-Function

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ABSTRACT

Let spt (n) denote the total number of appearances of the smallest part in each partition of n. In 1988, Garvan gave new combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11 in terms of a crank for weighted vector partitions. This paper shows how to generate the generating functions for spt(n), elaborately and also shows how to prove the relation among the terms spt (n) and. In 2008, Andrews stated Ramanujan- type congruences for the spt- function mod 5, 7 and 13. The new combinatorial interpretations of the spt- congruences mod 5 and 7 are given in this article. These are in terms of the spt- crank but for a restricted set of vector partitions. The proofs depend on relating the spt- crank with the crank of vector partitions.

Keywords: Crank, congruences, product notations, Ramanujan -type congruences, spt-function, vector partitions

10/12/2014 **Source of Support:** Nil, **No Conflict of Interest:** Declared.

How to Cite: Das S. 2014. **Congruence Properties of Andrews' SPT- Function** *ABC Journal of Advanced Research*, 3, 47-56.

INTRODUCTION

We give some related definitions of spt(n), vector partitions, $M_s(m,n)$, $M_s(m,t,n)$, and $(z,x)_\infty$. We discuss the generating function for spt (n) and prove the Theorem 1 in terms of $M_s(m,n)$ and also establish the relation among the terms spt (n), $M_s(m,n)$ and $\omega(\vec{\pi})$. In this paper how to prove the Theorems: $5/spt(5n+4)$, $7/spt(7n+5)$,and $13/spt(13n+6)$ with the help of examples. These Theorems are the combinatorial interpretations of Ramanujan's famous partition congruences mod 5, 7 and 13. The proofs of the Theorems 2, 3 and 4 depend on relating the spt- crank but for a restricted set of vector partitions.

SOME RELATED DEFINITIONS

spt(*n*): *spt*(*n*) is the total number of appearances of the smallest parts in all the partitions of *n*, like:

n		*spt*(*n*)
1	$\dot{1}$	1
2	$\dot{2}$, $\dot{1}+\dot{1}$	3
3	$\dot{3}$, $2+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$	5
4	$\dot{4}$, $3+\dot{1}$, $\dot{2}+\dot{2}$, $2+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}$	10
5	$\dot{5}$, $4+\dot{1}$, $3+\dot{2}$, $3+\dot{1}+\dot{1}$, $2+2+\dot{1}$, $2+\dot{1}+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}+\dot{1}$	14
…	…	…

Vector partitions[Garvan (2013)]:

Let, *P* denotes the set of partitions and *D* denotes the set of partitions into distinct parts. The set of vector partitions *V* is defined by the Cartesian product, $V = D \times P \times P$.

For a partition π, denote $S(\pi)$ as the smallest part in the partition with $S(\phi)=\infty$ for the empty partition. We denote the following subset of vector partitions,

$S = \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \le S(\pi_1) < \infty \text{ and } S(\pi_1) \le \min(S(\pi_2), S(\pi_3))\}$.

For $\vec{\pi} \in S$ we define the weight ω_1 by $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the crank $(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$ and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$, where $|\pi_j|$ is the sum of the parts of π_j.

$M_s(m,n)$: The number of vector partitions of *n* in *S* with crank *m* counted according to the weight ω is denoted by

$$M_s(m,n), \text{ so that } M_S(m,n) = \sum_{\vec{\pi} \in S, |\vec{\pi}|=n} \omega(\vec{\pi}).$$

$M_s(m,t,n)$: The number of vector partitions of *n* in *S* with crank congruent to *m* modulo *t* counted according to

the weight ω is denoted by $M_s(m,t,n)$, so that;

$$M_S(m,t,n) = \sum_{k=-\infty}^{\infty} M_S(kt+m,n) = \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}|=n \\ crank(\vec{\pi}) \equiv m (\text{mod}\, t)}} \omega(\vec{\pi})$$

and $(z;x)_\infty = (z)_\infty = \lim_{n \to \infty} (z;x)_n = \prod_{n=1}^{\infty} (1 - zx^{n-1}) = (1-z)(1-zx)(1-zx^2)\ldots$ where $|x|<1$.

GENERATING FUNCTION [GARVAN (1986)]

$$\sum_{n=1}^{\infty} spt(n) x^n = \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2 (x^{n+1};x)_\infty} = \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2 (1-x^{n+1})(1-x^{n+2})\ldots}$$

$$=\frac{x}{(1-x)^2(x^2;x)_\infty}+\frac{x^2}{(1-x)^2(x^3;x)_\infty}+\frac{x^3}{(1-x)^2(x^4;x)_\infty}+\dots$$

$$=x+3x^2+5x^3+10x^4+14x^5+26x^6+\dots$$
$$=spt(1)x+spt(2)x^2+spt(3)x^3+\dots$$

Theorem 1: $\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}M_S(m,n)z^m x^n=\sum_{n=1}^{\infty}\frac{x^n(x^{n+1};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}$

Proof: If $t\geq 1$ then, $\frac{x^t(x^{t+1};x)_\infty}{(zx^t;x)_\infty(z^{-1}x^t;x)_\infty}$

$$=\frac{x^t(1-x^{t+1})(1-x^{t+2})(1-x^{t+3})\dots}{(1-zx^t)(1-zx^{t+1})\dots(1-z^{-1}x^t)(1-z^{-1}x^{t+1})\dots}$$

$$=\left(\sum_{\substack{\pi_1\in D\\ S(\pi_1)=t}}(-1)^{\#(\pi_1)-1}x^{|\pi_1|}\right)\left(\sum_{\substack{\pi_2\in P\\ t\leq S(\pi_2)}}z^{\#(\pi_2)}x^{|\pi_2|}\right)\left(\sum_{\substack{\pi_3\in P\\ t\leq S(\pi_3)}}z^{-\#(\pi_3)}x^{|\pi_3|}\right)$$ [Andrews (1985)]

$$=\sum_{\substack{\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in S\\ S(\pi_1)=t}}\omega(\vec{\pi})\,z^{\text{crank}(\vec{\pi})}x^{|\vec{\pi}|}.$$

So that; $\sum_{n=1}^{\infty}\frac{x^n(x^{n+1};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}=\sum_{t=1}^{\infty}\sum_{\substack{\vec{\pi}\in S\\ S(\pi_1)=t}}\omega(\vec{\pi})\,z^{\text{crank}(\vec{\pi})}x^{|\vec{\pi}|}$

$$=\sum_{\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in S}\omega(\vec{\pi})\,z^{\text{crank}(\vec{\pi})}x^{|\vec{\pi}|}$$

$$=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}M_S(m,n)z^m x^n$$, [Andrews,etel (1988)].

Corollary 1: For $n\geq 1$, $spt(n)=\sum_{m=-\infty}^{\infty}M_S(m,n)=\sum_{\vec{\pi}\in S,|\vec{\pi}|=n}\omega(\vec{\pi})$

Proof: If z = 1 from above we get;

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}M_S(m,n)x^n=\sum_{\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in S}\omega(\vec{\pi})\,x^{|\vec{\pi}|}=\sum_{t=1}^{\infty}\sum_{\substack{\vec{\pi}\in S,|\vec{\pi}|=n\\ S(\pi_1)=t}}\omega(\vec{\pi})\,x^{|\vec{\pi}|}$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{n+1};x)_\infty}{(x^n;x)_\infty (x^n;x)_\infty} = \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2 (x^{n+1};x)_\infty} = \sum_{n=1}^{\infty} spt(n) x^n .$$

Equating the coefficient of x^n we get;

$$spt(n) = \sum_{\vec{\pi} \in S, |\vec{\pi}| = n} \omega(\vec{\pi}) = \sum_{m=-\infty}^{\infty} M_S(m,n)$$

i.e., $spt(n) = \sum_{m=-\infty}^{\infty} M_S(m,n) = \sum_{\vec{\pi} \in S, |\vec{\pi}| = n} \omega(\vec{\pi}).$

Theorem 2: $M_S(k,5,5n+4) = \dfrac{spt(5n+4)}{5}$, for $0 \le k \le 4$.

Proof: We prove Theorem 2 with an example. There is a table of the 16 vector partitions $\vec{\pi} \in S$ with $|\vec{\pi}| = 4$ as follows:

Table-1

Vector partitions of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (4, \phi, \phi)$	+1	0
$\vec{\pi}_2 = (3+1, \phi, \phi)$	–1	0
$\vec{\pi}_3 = (1, 3, \phi)$	+1	1
$\vec{\pi}_4 = (1, \phi, 3)$	+1	–1
$\vec{\pi}_5 = (2, 2, \phi)$	+1	1
$\vec{\pi}_6 = (2, \phi, 2)$	+1	–1
$\vec{\pi}_7 = (2+1, 1, \phi)$	–1	1
$\vec{\pi}_8 = (2+1, \phi, 1)$	–1	–1
$\vec{\pi}_9 = (1, 1+2, \phi)$	+1	2
$\vec{\pi}_{10} = (1, \phi, 1+2)$	+1	–2
$\vec{\pi}_{11} = (1, 1, 2)$	+1	0
$\vec{\pi}_{12} = (1, 2, 1)$	+1	0
$\vec{\pi}_{13} = (1, 1+1+1, \phi)$	+1	3
$\vec{\pi}_{14} = (1, \phi, 1+1+1)$	+1	–3
$\vec{\pi}_{15} = (1, 1+1, 1)$	+1	1
$\vec{\pi}_{16} = (1, 1, 1+1)$	+1	–1

From the table we get,

$M_S(0,5.4) = \omega(\vec{\pi}_1) + \omega(\vec{\pi}_2) + \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{12})$ =1–1+1+1= 2.

Similarly, $M_S(0,5,4) = M_S(1,5,4) = M_S(2,5,4) = M_S(3,5,4) =$

$M_S(4,5,4) = 2 = \frac{spt(4)}{5}$.

Hence, $M_S(k,5,5n+4) = \frac{spt(5n+4)}{5}$, for $0 \le k \le 4$. Hence, the Theorem.

We can find the following relations from above table:

$M_S(0,5,4)$ =+1–1+1+1=2,

$M_S(1,5,4)$ =+1+1–1+1=2,

$M_S(2,5.4) = M_S(-3,5,4)$=+1+1=2,

$M_S(3,5,4) = M_S(-2,5,4)$=+1+1=2,

$M_S(4,5.4) = M_S(-1,5,4)$=+1+1–1+1=2.

So that we can see that, $M_S(m,n) \ge 0$ for all m and n.

$M_S(m,n) = M_S(-m,n)$ and $M_S(m,t,n) = M_S(t-m,t,n)$.

Theorem 3: $M_S(k,7,7n+5) = \frac{spt(7n+5)}{7}$, for $0 \le k \le 6$.

Proof: We prove the Theorem 3 with an example. There is a table of the 32 vector partitions $\vec{\pi} \in S$ with $|\vec{\pi}| = 5$ as follows:

Table-2

Vector partitions of 5	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (5, \phi, \phi)$	+1	0
$\vec{\pi}_2 = (1,4,\phi)$	+1	1
$\vec{\pi}_3 = (1,\phi,4)$	+1	–1
$\vec{\pi}_4 = (2,3,\phi)$	+1	1
$\vec{\pi}_5 = (2,\phi,3)$	+1	–1
$\vec{\pi}_6 = (1,3+1,\phi)$	+1	2
$\vec{\pi}_7 = (1,\phi,3+1)$	+1	–2
$\vec{\pi}_8 = (1,3,1)$	+1	0
$\vec{\pi}_9 = (1,1,3)$	+1	0
$\vec{\pi}_{10} = (1,2,2)$	+1	0

$\vec{\pi}_{11} = (1, 2+2, \phi)$	+1	2
$\vec{\pi}_{12} = (1, \phi, 2+2)$	+1	–2
$\vec{\pi}_{13} = (1, 1+1+2, \phi)$	+1	3
$\vec{\pi}_{14} = (1, \phi, 1+1+2)$	+1	–3
$\vec{\pi}_{15} = (1, 1+1, 2)$	+1	1
$\vec{\pi}_{16} = (1, 2, 1+1)$	+1	–1
$\vec{\pi}_{17} = (1, 1+2, 1)$	+1	1
$\vec{\pi}_{18} = (1, 1, 1+2)$	+1	–1
$\vec{\pi}_{19} = (1, 1+1+1+1, \phi)$	+1	3
$\vec{\pi}_{20} = (1, \phi, 1+1+1+1)$	+1	–3
$\vec{\pi}_{21} = (1, 1+1, 1+1)$	+1	0
$\vec{\pi}_{22} = (1, 1+1+1, 1)$	+1	2
$\vec{\pi}_{23} = (1, 1, 1+1+1)$	+1	–2
$\vec{\pi}_{24} = (1+3, 1, \phi)$	–1	+1
$\vec{\pi}_{25} = (1+3, \phi, 1)$	–1	–1
$\vec{\pi}_{26} = (1+4, \phi, \phi)$	–1	0
$\vec{\pi}_{27} = (2+3, \phi, \phi)$	–1	0
$\vec{\pi}_{28} = (2+1, 2, \phi)$	–1	1
$\vec{\pi}_{29} = (2+1, \phi, 2)$	–1	–1
$\vec{\pi}_{30} = (2+1, 1, 1)$	–1	0
$\vec{\pi}_{31} = (2+1, 1+1, \phi)$	–1	2
$\vec{\pi}_{32} = (2+1, \phi, 1+1)$	–1	–2

From the above table we get,

$M_S(0,7,5) =$ +1+1+1+1+1–1–1–1–1=2

$M_S(1,7,5) =$ +1+1–1–1+1+1=2

$M_S(2,7,5) =$ +1+1+1–1=2

$M_S(3,7,5) = M_S(-4,7,5) =$ +1+1=2

$M_S(4,7,5) = M_S(-3,7,5) =$ +1+1=2

$M_S(5,7,5) = M_S(-2,7,5) =$ +1+1+1–1=2

$M_S(6,7,5) = M_S(-1,7,5) = +1+1-1-1+1+1=2$

So that, $M_S(0,7,5) = M_S(1,7,5) = M_S(2,7,5) = M_S(3,7,5) =$

$M_S(4,7,5) = M_S(5,7,5) = M_S(6,7,5) = 2 = \dfrac{spt(5)}{7}$.

Hence, $M_S(k,7,7n+5) = \dfrac{spt(7n+5)}{7}$, for $0 \le k \le 6$. Hence, the Theorem.

Theorem 4: $spt(13n+6) \equiv 0 \pmod{13}$.

Proof: We prove the Theorem 4 with an example. There is a table of the 64 vector partitions $\vec{\pi} \in S$ with $|\vec{\pi}| = 6$ as follows:

Table-3

Vector partitions of 6	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (6,\phi,\phi)$	+1	0
$\vec{\pi}_2 = (1+5,\phi,\phi)$	–1	0
$\vec{\pi}_3 = (1,5,\phi)$	+1	+1
$\vec{\pi}_4 = (1,\phi,5)$	+1	–1
$\vec{\pi}_5 = (2+4,\phi,\phi)$	–1	0
$\vec{\pi}_6 = (2,\phi,4)$	+1	–1
$\vec{\pi}_7 = (2,4,\phi)$	+1	1
$\vec{\pi}_8 = (1,1+4,\phi)$	+1	2
$\vec{\pi}_9 = (1,\phi,1+4)$	+1	–2
$\vec{\pi}_{10} = (1,1,4)$	+1	0
$\vec{\pi}_{11} = (1,4,1)$	+1	0
$\vec{\pi}_{12} = (1+4,1,\phi)$	–1	+1
$\vec{\pi}_{13} = (1+4,\phi,1)$	–1	–1
$\vec{\pi}_{14} = (3,3,\phi)$	+1	1
$\vec{\pi}_{15} = (1,2,3)$	+1	–1
$\vec{\pi}_{16} = (1,2,3)$	+1	0
$\vec{\pi}_{17} = (1,3,2)$	+1	0
$\vec{\pi}_{18} = (1,2+3,\phi)$	+1	2
$\vec{\pi}_{19} = (1,\phi,2+3)$	+1	–2

$\vec{\pi}_{20} = (1+2, \phi, 3)$	–1	–1
$\vec{\pi}_{21} = (1+2, 3, \phi)$	–1	1
$\vec{\pi}_{22} = (1+3, \phi, 2)$	–1	–1
$\vec{\pi}_{23} = (1+3, 2, \phi)$	–1	1
$\vec{\pi}_{24} = (1, 1+3, 1)$	+1	1
$\vec{\pi}_{25} = (1, 1, 1+3)$	+1	–1
$\vec{\pi}_{26} = (1+3, 1+1, \phi)$	–1	2
$\vec{\pi}_{27} = (1+3, \phi, 1+1)$	–1	–2
$\vec{\pi}_{28} = (1+3, 1, 1)$	–1	0
$\vec{\pi}_{29} = (2, 2, 2)$	+1	0
$\vec{\pi}_{30} = (2, 2+2, \phi)$	+1	2
$\vec{\pi}_{31} = (2, \phi, 2+2)$	+1	–2
$\vec{\pi}_{32} = (1+2, 2, 1)$	–1	0
$\vec{\pi}_{33} = (1+2, 1, 2)$	–1	0
$\vec{\pi}_{34} = (1+2, 1+2, \phi)$	–1	2
$\vec{\pi}_{35} = (1+2, \phi, 1+2)$	–1	–2
$\vec{\pi}_{36} = (1, 1+1, 1+1+1)$	+1	–1
$\vec{\pi}_{37} = (1, 1+1+1, 1+1)$	+1	1
$\vec{\pi}_{38} = (1, 1+1+1+1, 1)$	+1	3
$\vec{\pi}_{39} = (1, 1, 1+1+1+1)$	+1	–3
$\vec{\pi}_{40} = (1, 1+1+1+1+1, \phi)$	+1	5
$\vec{\pi}_{41} = (1, \phi, 1+1+1+1+1)$	+1	–5
$\vec{\pi}_{42} = (1, 1+1+1, 2)$	+1	2
$\vec{\pi}_{43} = (1, 1+1+1+2, \phi)$	+1	4
$\vec{\pi}_{44} = (1, \phi, 1+1+1+2)$	+1	–4
$\vec{\pi}_{45} = (1, 2, 1+1+1)$	+1	–2
$\vec{\pi}_{46} = (1, 1+1, 1+2)$	+1	0
$\vec{\pi}_{47} = (1, 1+2, 1+1)$	+1	0
$\vec{\pi}_{48} = (1, 1, 1+1+2)$	+1	–2
$\vec{\pi}_{49} = (1, 1+1+2, 1)$	+1	2

$\vec{\pi}_{50} = (1+2, 1+1+1, \phi)$	–1	3
$\vec{\pi}_{51} = (1+2, \phi, 1+1+1)$	–1	–3
$\vec{\pi}_{52} = (1+2, 1+1, 1)$	–1	1
$\vec{\pi}_{53} = (1+2, 1, 1+1)$	–1	–1
$\vec{\pi}_{54} = (1, 1+2+2, \phi)$	+1	3
$\vec{\pi}_{55} = (1, \phi, 1+2+2)$	+1	–3
$\vec{\pi}_{56} = (1, 1+2, 2)$	+1	1
$\vec{\pi}_{57} = (1, 2, 1+2)$	+1	–1
$\vec{\pi}_{58} = (1, 2+2, 1)$	+1	1
$\vec{\pi}_{59} = (1, 1, 2+2)$	+1	–1
$\vec{\pi}_{60} = (1, 1+1+3, \phi)$	+1	3
$\vec{\pi}_{61} = (1, \phi, 1+1+3)$	+1	–3
$\vec{\pi}_{62} = (1, 1+1, 3)$	+1	1
$\vec{\pi}_{63} = (1, 3, 1+1)$	+1	–1
$\vec{\pi}_{64} = (1+2+3, \phi, \phi)$	+1	0

From the table we get; $M_S(0,13,6)$ = +1–1–1+1+1+1+1–1+1–1–1+1+1+1 = 4,

$M_S(1,13,6)$ = +1+1–1+1–1–1+1+1–1+1+1+1 = 4,

$M_S(2,13,6)$ = +1+1–1+1–1+1+1 = 3,

$M_S(3,13,6)$ = +1–1+1+1 = 2,

$M_S(4,13,6)$ = +1= 1,

$M_S(5,13,6)$ = +1= 1,

$M_S(6,13,6)$ = 0,

$M_S(7,13,6)$ = 0,

$M_S(8,13,6) = M_S(-5,13,6)$ = +1 = 1,

$M_S(9,13,6) = M_S(-4,13,6)$ = +1 = 1,

$M_S(10,13,6) = M_S(-3,13,6)$ = +1–1+1+1 = 2,

$M_S(11,13,6) = M_S(-2,13,6)$ = +1+1–1+1–1+1+1 = 3,

$M_S(12,13,6) = M_S(-1,13,6)$ = +1+1–1+1–1–1+1+1 –1+1+1+1= 4.

$$\therefore \sum_{m=-5}^{5} M_s(m,13,6) = \sum_{m=0}^{12} M_s(m,13,6) = spt(13n+6) = 26, \text{ where } n = 0.$$

i.e., $spt(13n+6) \equiv 0 \pmod{13}$. Hence the Theorem.

CONCLUSION

In this study we have discussed the set of vector partitions and have discussed the generating function for spt (n) and also have established the generating function for $M_s(m,n)$. We have shown a relation among the terms spt (n), $M_s(m,n)$, and $\omega(\vec{\pi})$ and have satisfied the Theorems 2, 3, and 4 with the help of examples.

ACKNOWLEDGMENT

It is a great pleasure to express my sincerest gratitude to my respected teacher Professor Md. Fazlee Hossain, Department of Mathematics, University of Chittagong, Bangladesh.

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