BOUNDS ON NON-SYMMETRIC DIVERGENCE

R. N. SARASWAT
Department of Mathematics and Statistics, Manipal University, Jaipur,
INDIA-303007, E-mail: sarswatrn@gmail.com

Abstract: Inequalities are playing a fundamental role in the area of Information Theory and Statistics. Large numbers of Mathematicians like as Taneja , Pranesh Kumar Dragomir etc. have been studied application of Inequalities in different branches of the pure and applied mathematics, physics, computer science etc. In this research article, we shall consider inequalities among information divergence measure and Hellinger discrimination. We shall propose a new non-symmetric information divergence measure and bounds of new information divergence measure are also considered in this research article.

Keywords: - Csiszar's f-divergence measure, Hellinger Discrimination, Bhattacharya divergence measure, relative information of type's, information inequalities, Kullback-Leibler divergence measure and numerical illustrations etc.

1. INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ...p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2 \tag{1.1}$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures exists in the literature on information theory. Csiszar [2] & [3] introduced a generalized measure of information using f-divergence measure is given by

$$I_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \tag{1.2}$$

where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function and $P, Q \in \Gamma_n$.

Here we list some existing divergence measures which are in the category of f-divergence measures, together with the suitable generating function f.

- **Hellinger Discrimination** [4]:-

$$h(P,Q) = [1 - B(P,Q)] = \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \qquad (1.3)$$

where $B(P,Q) = \sum_{i=1}^{n}\sqrt{p_i q_i}$ is known as Bhattacharya divergence measure [1]

- **Relative information of type s** [8]

The following measures and particular cases are introduced [8]

$$\Phi_s(P,Q) = \begin{cases} {}^2K_s(P,Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], s \neq 0,1 \\ D(Q,P) = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right), & s = 0 \\ D(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases} \qquad (1.4)$$

In whole paper, in the section 2, we have introduced information inequalities. New non-symmetric information divergence measure has discussed in section 3. Bounds of new non symmetric divergence measure have also studied in section 4.

2. NEW INFORMATION INEQUALITY

The following propositions are one of the results of the theorem is give in [8] and similar line to [5], [6] & [7] and proposition 2.2 is the particular case for $s = 1/2$ of proposition 2.1.

Proposition 2.1:- Let $f:(0,\infty) \to \mathbf{R}$ be a mapping which is normalized i.e. $f(1) = 0$ and satisfies the assumptions.

(i) f is twice differentiable on (r, R) .where $0 \le r \le 1 \le R \le \infty$

(ii) there exist the real Constants m , M such that $m < M$

$$m \le t^{2-s} f''(t) \le M, \forall t \in (r,R), s \in \mathbf{R} \qquad (2.1)$$

If $P,Q \in \Gamma_n$ are discrete probability distributions satisfying assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty, \ \forall i \in \{1,2,3,.........n\} \qquad (2.2)$$

then we have the inequality

$$m\,\Phi_s(P,Q) \le I_f(P,Q) \le M\,\Phi_s(P,Q) \qquad (2.3)$$

Where $\Phi_s(P,Q)$ is given by (1.2), for $s = 1/2$ respectively.

Proposition 2.2:- Let $f:(0,\infty)\to\mathbf{R}$ is normalized i.e. $f(1)=0$ and satisfies the assumptions.

(i) f is twice differentiable on (r, R) .where $0\le r\le 1\le R\le\infty$

(ii) There exist Constant m, M such that $m<M$

$$m\le t^{3/2} f''(t)\le M, \forall t\in(r,R) \tag{2.4}$$

If $P,Q\in\Gamma_n$ are discrete probability distributions satisfying assumption

$$0<r\le\frac{p_i}{q_i}\le R<\infty,\ \forall i\in\{1,2,3,\dots\dots\dots n\}$$

Then we have the inequality

$$4m\ h(P,Q)\le I_f(P,Q)\le 4M\ h(P,Q) \tag{2.5}$$

In view of proposition (4.1) we states the following results

3. INFORMATION DIVERGENCE MEASURE

In this section we introduce a new information divergence measure which is the category of Csiszar's f-divergence measure. Let us consider the function $f:(0,\infty)\to\mathbf{R}$

$$f(t)=\frac{(t^2-1)^2}{t},\quad f'(t)=\frac{3t^4-2t^2-1}{t^2},\ f''(t)=\frac{6t^4+2}{t^3}>0,\forall t>0 \tag{3.1}$$

Hence function f (t) is convex from equation 3.1 and figure 3.1, f (1) =0 i.e. normalized.

Applying Csiszar's f-divergence properties on (1.2), then we get

$$I_f(P,Q)=\sum_{i=1}^{n}\frac{(p_i^2-q_i^2)^2}{q_i^2\,p_i}=4\sum_{i=1}^{n}\left(1\Big/\frac{2p_iq_i}{p_i+q_i}\right)\frac{(p_i+q_i)}{2}\frac{(p_i-q_i)^2}{q_i}=N(P,Q) \tag{3.2}$$

Where " $N(P,Q)$ " is made up the combination of Harmonic, Arithmetic and χ^2-divergence measure. Above new information divergence measure are represented in the figure 3.1. It is clear that from the figure 3.1 and 3.2 the convex function $f(t)$ gives a steeper slope. Further $f(1)=0$, so that $N(P,P)=0$ and the convexity of the function $f(t)$ ensure that the measure (3.2) is non-negative.

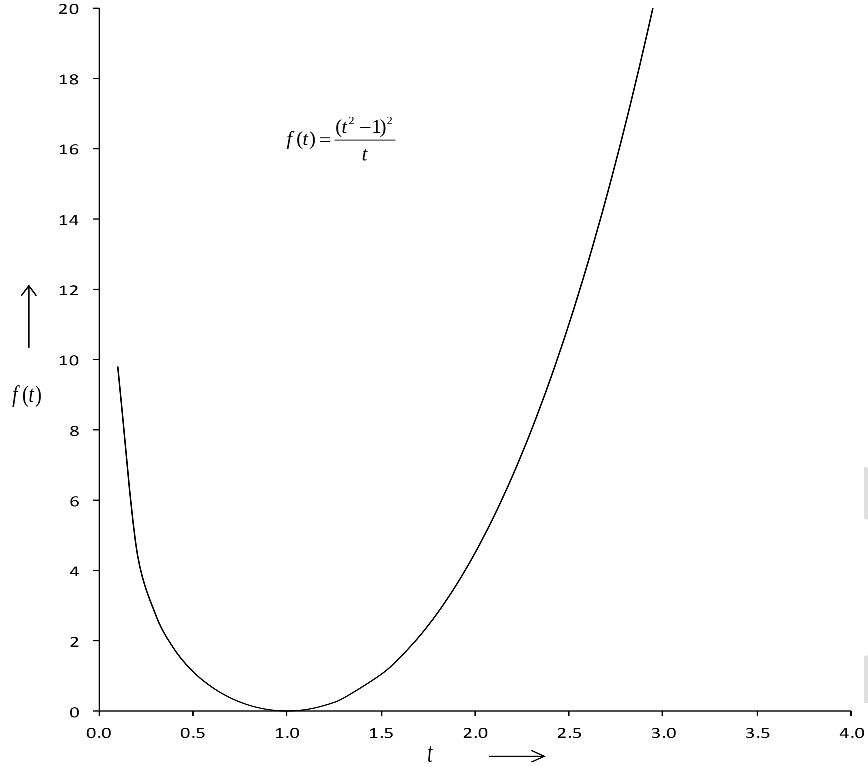

Figure 3.1

Figure 3.1 represent convexity of new non symmetric information divergence measure

In following section we shall consider some results using proposition (2.1) and the measure $N(P,Q)$ given in equation (3.2) which results are similar line to [4], [6] & [7].

4. RESULTS

In this section we shall consider the bounds of new information divergence measure in terms of Hellinger discrimination.

Result: 4.1: - Let $P,Q\in\Gamma_n$ and $s=1/2$ .Let there exists r, R such that r < R and $0<r\le\frac{p_i}{q_i}\le R<\infty\ \forall\ i\in\{1,2,3,.........n\}$

(i) If $0<r<0.67$

$$23.4\,h(P,Q)\le\ N(P,Q)\le 4\max\left\{\frac{6r^4+2}{r^{3/2}},\frac{6R^4+2}{R^{3/2}}\right\}h\left(P,Q\right) \tag{4.1}$$

(ii) If $0.67<r<\infty$

$$\frac{3r^4+1}{r^{3/2}}h(P,Q)\le\frac{1}{8}N(P,Q)\le\frac{3R^4+1}{R^{3/2}}h\left(P,Q\right) \tag{4.2}$$

Proof:-

From equations (3.1), (3.2) & (2.4), we get

$$g(t) = t^{3/2} f''(t) = \frac{6t^4 + 2}{t^{3/2}} > 0, \forall t > 0$$

$$g(t) = 6t^{5/2} + \frac{2}{t^{3/2}} > 0, \forall t > 0,$$

$$g'(t) = 15t^{3/2} - \frac{3}{t^{5/2}} = 0$$

$$g''(t) = \frac{45}{2} t^{1/2} + \frac{15}{2} t^{-7/2}$$

$$g'(t) = 15t^{3/2} - \frac{3}{t^{5/2}} = 0 \Rightarrow t^4 - \frac{1}{5} = 0 \Rightarrow t = \left(\frac{1}{5}\right)^{1/4} = 0.67$$

It is clear that g (t) is monotonic decreasing on [0, .67) and monotonic increasing on $[0.67, \infty)$.

Also function g has minimum realized at $t_0 = 0.67$

$$g''(t) = \frac{45}{2}(.67)^{1/2} + \frac{15}{2}(.67)^{-3/2} \text{ (Positive)}$$

and

$$\inf_{t \in (0,\infty)} g(t) = g(.67) = 6(.67)^{5/2} + \frac{2}{(.67)^{3/2}} = 2.20 + 3.7 = 5.9$$

We have two cases:

(i) If $0 < r < 0.67$ then

$$m = \inf_{t \in (r.R)} g(t) = g(.67) = 6(.67)^{5/2} + \frac{2}{(.67)^{3/2}} = 5.85 \quad (4.3)$$

$$M = \sup_{t \in (r,R)} g(t) = \max\{g(r), g(R)\} = \max\left\{\frac{6r^4 + 2}{r^{3/2}}, \frac{6R^4 + 2}{R^{3/2}}\right\} \quad (4.4)$$

(ii) If $0.67 < r < \infty$

$$m = \inf_{t \in (r,R)} g(t) = \frac{6r^4 + 2}{r^{3/2}} \quad M = \sup_{t \in (r,R)} g(t) = \frac{6R^4 + 2}{R^{3/2}} \quad (4.5)$$

Equation (2.4) of proposition (2.2) using equation (3.2), (4.3), (4.4) & (4.5) gives the results (4.1) & (4.2).

5. NUMERICAL STUDY

Example 5.1

Let P be the binomial probability distribution for the random valuable X with parameter (n=8 p=0.5) and Q its approximated normal probability distribution. The following table have also discussed [8].

Table 5.1 Binomial Probability Distribution (n=8 p=0.5)

x	0	1	2	3	4	5
p (x)	0.004	0.031	0.109	0.219	0.274	0.219
q (x)	0.005	0.030	0.104	0.220	0.282	0.220
p(x)/q(x)	0.774	1.042	1.0503	0.997	0.968	0.997

It is noted that $r = 0.77$ and $R = 1.05$. Here we shall discuss the numerical bounds of new information divergence measure in terms of Hellinger discrimination. From equation (4.1) and (4.2) and using the table of Binomial distribution where R and r are the lower and upper bounds then we get

(i) If $0 < r < .67$.

$$23.4\,h(P,Q) \le N(P,Q) \le 4\max\left\{\frac{6r^4+2}{r^{3/2}}, \frac{6R^4+2}{R^{3/2}}\right\} h(P,Q)$$

$$23.4\,h(P,Q) \le N(P,Q) \le 4\max\left\{\frac{4.155355391}{(.774179933)^{3/2}} = .681181532, \frac{9.302210714}{1.076437122} = 8.4166591\right\} h(P,Q)$$

$$23.4\,h(P,Q) \le N(P,Q) \le 4 \times 8.4166591\,h(P,Q)$$

$$23.4\,h(P,Q) \le N(P,Q) \le 34.5666636\;h(P,Q)$$

(ii) If $.67 < r < \infty$

$$\frac{3r^4+1}{r^{3/2}} h(P,Q) \le \frac{1}{8} N(P,Q) \le \frac{3R^4+1}{R^{3/2}} h(P,Q)$$

$$0.681181532\,h(P,Q) \le \frac{1}{8} N(P,Q) \le 34.5666636\;h(P,Q)$$

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