On the Derivation of Some Reduction Formula through Tabular Integration by Parts

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Date Received: December 29, 2014; Date Revised: February 15, 2015

Abstract *– The study aimed to expose the application of the algorithm of the Tabular Integration by Parts (TIBP in the derivation of some reduction formula involving integrals of product of elementary functions. In the study, the existing algorithm of tabular integration by parts (TIBP) was modified by forming a table consisting of three columns namely: S column where alternating plus and minus signs are written, D column where f(x) and its successive derivatives are listed, and I column where g(x) and its successive antiderivative/integrals are listed. The modified algorithm of TIBP was then used to evaluate the reduction formula involving the integrals of the forms:* $\int x^n \sin axdx$, $\int x^n \cos axdx$, $\int x^n \sinh axdx$, $\int x^n \cosh axdx$, $\int x^n e^{ax} dx$ *,and* $\int x^n (\ln x)^m dx$ *where n, m and a are any real number.*

Keywords *– elementary functions, integrals, reduction formula, tabular integration by parts*

INTRODUCTION

There are several techniques of integration which can be used to evaluate the integrals of elementary functions. One of those techniques is the integration by parts which can be used for simplifying integrals of the form $\int f(x)\, g(x)dx$ in which *f(x)* can be differentiated repeatedly and *g(x)* can be integrated repeatedly without difficulty. It introduces us to a technique of great adaptability and wide applications. As a method within a major branch of calculus, integration-by-parts by itself can provide more than one road into that precise answer. The integration by parts formula given by $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$ can be written as $\int udv = uv - \int vdu$ [1] – [3].

An acronym that is very helpful to remember when using integration by parts is LIATE. The rule was first mention in a paper of Kasube [4]. The word LIATE stands for Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential. Based from the rule, if there is a combination of two of these types of functions in the original integral, choose for "*u*" the type that appear first in the LIATE and "*dv*" is whatever is left, that is, the part that appears second in the LIATE. Basically, the function which is to be "*dv*" comes later in the list, that is, functions that have easier anti-derivatives than the functions before them.

After several applications of the LIATE Rule in evaluating the integrals of products of elementary functions, the researcher suggested that the rule can be modified by including hyperbolic functions. The hyperbolic functions can be included from the original LIATE Rule and make it LIATHE Rule. The following are the reasons for placing it in between the Trigonometric and Exponential functions: (1) hyperbolic functions behave like that of trigonometric functions when differentiating and integrating them, and (2) hyperbolic functions can be expressed in the same form as exponential functions. From these reasons, it follows from the above rationale given by Kasube, the hyperbolic functions like the trigonometric and exponential functions, bring up the rear, since their differentiation and antidifferentiation results in similar functions.

In another paper, Nicol [5] commented that nearly every calculus textbook that have been encountered in the past several years uses the identities to evaluate integrals of certain form. Most students balk in anticipation of more formulas to memorize. These integrals are typically found in the section of a text dealing with integrating powers of trigonometric functions, which follows the section on integration by parts. It can be contended that these integrals should be done by repeated (iterated) integration by parts. Finally, for a student who has been taught tabular integration by parts, the calculation will consider the integral that is evaluated without the use of trigonometric identities and, preferably, in terms of the arguments of the trigonometric functions found in the original problem. Incidentally, checking the example presented and a few others by differentiation may prompt some to notice the forms that appear as antiderivatives and thereby to sense the possibility of yet another method: the undetermined coefficients.

Tabular integration by parts was featured in the article of Gillman, L. [6] which commented on the engaging article by David Horowitz [7]. The method is based on iterating the diagram

u ———— v' The integral of this product

equals this product

u' ———— v minus the integral of this product,

and can be ended at any stage. Tabular integration by parts streamlines the integrations and also makes proofs of operational properties more elegant and accessible.

The notes of Kowalski [8] commented that the integration by parts formula can be difficult to apply repeatedly; it takes a lot of space to write down and it's easy to make distribution error. Fortunately, there is a purely mechanical procedure for performing integration by parts without writing down so much called Tabular Integration by Parts. The notes presented theorem and tables to illustrate the process.

In an article by Switkes [9], the technique of Integration by Parts as an integration rule corresponding to the Product Rule for differentiation was introduced. The derivation of a Quotient Rule Integration by Parts formula was presented, applying the resulting integration formula to an example, and discuss reasons why this formula does not appear in calculus texts.

Column Integration or tabular integration by parts was cited in an article of Dence [10] as a method that calls for two columns of functions, labeled D and I. To determine the integral of the form $\int f(x)g'(x)dx$, *f(x)* is placed under D column, then successively differentiate it, while *g'(x)* is placed under I column and is then followed by successive antiderivatives. Then, in many cases, a series of cross-multiplications, with alternating signs converges to the integral $\int f(x)g'(x)dx$. In most applications *f(x)* is a polynomial, so the D column ends after finitely many steps and column integration yields $\int f(x)g'(x)dx$. If *f(x)* is not a polynomial, one can always truncate the process at any level and obtain a remainder term defined as the integral of the product of the two terms directly across from each other, namely $f^{(k)}(x)$ and $g^{(-k+1)}(x)$. Then, the infinite series would converge if the remainder tends to zero.

This integration by parts is a technique for evaluating an integral of a product of functions taught to college students taking Integral Calculus course. The goal of the integration by parts is to go from an integral $\int u\,dv$ that one finds it difficult to evaluate to an integral $\int v\,du$ that can be easily evaluated. Some integrals require repeated applications of integration by parts, which can become tedious and troubling for students, so the tabular method was developed from the integration by parts technique. It shortcuts the process and organizes work in table. Unfortunately this method is not commonly taught or utilized. Many popular calculus textbooks do not even mention the tabular method and, if they do, it is usually brief. It is for these reasons that prompted the researcher to conduct a study that will examine and explore the integrals of a product of functions normally beyond the capabilities of the tabular method in an attempt to expand its applications.

The general objective of the study is to expose the applications of Tabular Integration by Parts in deriving some reduction formula involving integrals of product of elementary functions. Specifically, it aims to

1. Modify the Horowitz's algorithm of the Tabular Integration by Parts.
2. Derive some reduction formula using Tabular Integration by parts.

The study is very significant to the students for additional insights of the use of Tabular Integration by Parts as an alternative, efficient and elegant way of finding the integrals of product of elementary functions are provided. It aims to help the readers appreciate the techniques of integration in a more interesting way.

METHODS

This study as a pure research is descriptive and expository in nature and like any other researches of this type it would follow its own form and style. As a descriptive research, this would be more than a description of the concepts in mathematics for it would expose and clarify the formulae and propositions concerned with the study.

RESULTS AND DISCUSSION

A. Modified Algorithm of the Tabular Integration by Parts

The algorithm of the Tabular Integration by Parts (TIBP) given by Horowitz [7] was modified. The modified algorithm can be shown in Table 1.

In column 1 (the S Column) of the table write alternating plus and minus signs. In column 2 (the D column) list f(x) and its successive derivatives. In column 3 (the I column) list g(x) and its successive antiderivatives. Form successive terms by multiplying each entry in the S and D columns by the entry in the I

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column that lies below it. The resulting sum of these terms is the integral.

Table 1. Table for evaluating $\int f(x)g(x)dx$.

Sign	Derivative	Integral
+	$f(x)$	$g(x)$
–	$f^1(x)$	$g^{[1]}(x)$
+	$f^2(x)$	$g^{[2]}(x)$
–	$f^3(x)$	$g^{[3]}(x)$
⋮	⋮	⋮
$(-1)^n$	$f^n(x)$	$g^{[n]}(x)$
$(-1)^{n+1}$	$f^{n+1}(x)$	$g^{[n+1]}(x)$

If f(x) is a polynomial, then there will be only a finite number of terms to sum. Otherwise the process may be truncated at any level by forming a remainder term defined as the product of the integral of the entries in S and D columns and the entry in the I column that lies directly across from it. In this table, $f^i(x)$ denotes the i^{th} derivative of the function $f(x)$ while $g^{[i]}(x)$ denotes the i^{th} integration (antiderivative) of $g(x)$. This algorithm can be written symbolically as

$$\int f(x)g(x)dx = f(x)g^{[1]}(x) - f^1(x)g^{[2]}(x) + f^2(x)g^{[3]}(x) - \dots + (-1)^n f^n(x)g^{[n+1]}(x) + (-1)^{n+1}\int f^{n+1}(x)g^{[n+1]}(x)dx$$

$$= \sum_{i=0}^{n}(-1)^i f^i(x)g^{[i+1]}(x) + (-1)^{n+1}\int f^{n+1}(x)g^{[n+1]}(x)dx \quad (1)$$

B. Reduction Formula Involving the Integrals of Product of Elementary Functions

The algorithm of the Tabular Integration by Parts (TIBP) can be used to derive the following reduction formula:

Example 1. Derive the formula:

$$\int x^n \sin axdx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos axdx \quad (2)$$

where *n* and *a* are any real numbers.

Derivation:

Applying the LIATHE Rule, let $u = x^n$ and $dv = \sin axdx$. Using the algorithm of the TIBP, Table 1 can be formed.

Table 2. TIBP for evaluating $\int x^n \sin axdx$, $n, a \in \Re$

Sign	Derivative	Integral
+	x^n	$\sin ax$
-	nx^{n-1}	$-\frac{1}{a}\cos ax$

The resulting integral from Table 2 is

$$\int x^n \sin axdx = (x^n)(-\frac{\cos ax}{a}) - \int (nx^{n-1})(-\frac{1}{a}\cos ax)dx$$

which can be simplified as

$$\int x^n \sin axdx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos axdx$$

Example 2. Derive the formula:

$$\int x^n \cos axdx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin axdx \quad (3)$$

where *n* and *a* are any real numbers.

Derivation:

Applying the LIATHE Rule, let $u = x^n$ and $dv = \cos axdx$. Using the algorithm of the TIBP, Table 3 can be formed.

Table 3. TIBP for evaluating $\int x^n \cos axdx$, $n, a \in \Re$

Sign	Derivative	Integral
+	x^n	$\cos ax$
-	nx^{n-1}	$\frac{1}{a}\sin ax$

The resulting integral from Table 3 is

$$\int x^n \cos axdx = (x^n)(\frac{\sin ax}{a}) - \int (nx^{n-1})(\frac{1}{a}\sin ax)dx$$

which can be simplified as

$$\int x^n \cos axdx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin axdx$$

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Example 3. Derive the formula:

$$\int x^n \sinh ax dx = \frac{x^n \cosh ax}{a} - \frac{n}{a}\int x^{n-1} \cosh ax dx \qquad (4)$$

where *n* and *a* are any real numbers.

Derivation:

Applying the LIATHE Rule, let $u = x^n$ and $dv = \sinh ax dx$. Using the algorithm of the TIBP, Table 4 can be formed.

Table 4. TIBP for evaluating $\int x^n \sin ax dx$, $n, a \in \Re$

Sign	Derivative	Integral
+ →	x^n ↘	$\sinh ax$
- →	nx^{n-1} →	$\frac{1}{a}\cosh ax$

The resulting integral from Table 4 is

$$\int x^n \sinh ax dx = (x^n)(\frac{\cosh ax}{a}) - \int (nx^{n-1})(\frac{1}{a}\cosh ax) dx$$

which can be simplified as

$$\int x^n \sinh ax dx = \frac{x^n \cosh ax}{a} - \frac{n}{a}\int x^{n-1} \cosh ax dx$$

Example 4. Derive the formula:

$$\int x^n \cosh ax dx = \frac{x^n \sinh ax}{a} - \frac{n}{a}\int x^{n-1} \sinh ax dx \qquad (5)$$

where *n* and *a* are any real numbers.

Derivation:

Applying the LIATHE Rule, let $u = x^n$ and $dv = \cosh ax dx$. Using the algorithm of the TIBP, Table 5 can be formed.

Table 5. TIBP for evaluating $\int x^n \cosh ax dx$, $n, a \in \Re$

Sign	Derivative	Integral
+ →	x^n ↘	$\cosh ax$
- →	nx^{n-1} →	$\frac{1}{a}\sinh ax$

The resulting integral from Table 5 is

$$\int x^n \cosh ax dx = (x^n)(\frac{\sinh ax}{a}) - \int (nx^{n-1})(\frac{1}{a}\sinh ax) dx$$

which can be simplified as

$$\int x^n \cosh ax dx = \frac{x^n \sinh ax}{a} - \frac{n}{a}\int x^{n-1} \sinh ax dx$$

Example 5. Derive the formula:

$$\int x^n e^{ax} dx = \frac{1}{a}[x^n e^{ax} - n\int x^{n-1} e^{ax} dx] \qquad (6)$$

where *n* and *a* are any real numbers.

Derivation:

Applying the LIATHE Rule, let $u = x^n$ and $dv = e^{ax}dx$. Using the algorithm of the TIBP, Table 6 can be formed.

Table 6. TIBP for evaluating $\int x^n e^{ax} dx$, $n, a \in \Re$

Sign	Derivative	Integral
+ →	x^n ↘	e^{ax}
- →	nx^{n-1} →	$\frac{1}{a}e^{ax}$

The resulting integral from Table 6 is

$$\int x^n e^{ax} dx = x^n \frac{1}{a} e^{ax} - \int nx^{n-1} \frac{1}{a} e^{ax} dx$$

which can be simplified as

$$\int x^n e^{ax} dx = \frac{1}{a}[x^n e^{ax} - n\int x^{n-1} e^{ax} dx]$$

Example 6. Derive the formula:

$$\int x^n (\ln x)^m dx = \frac{x^{n+1}}{n+1}(\ln x)^m - \frac{m}{n+1}\int (x^n)(\ln x)^{m-1} dx \qquad (7)$$

where n and m are any real numbers:

Derivation:

Applying the LIATE rule, let

$u = (\ln x)^m$ and $dv = x^n dx$

Using the algorithm of the TIBP, Table 7 can be formed.

Table 7. TIBP for evaluating $\int x^n (\ln x)^m dx$, $n, m \in \Re$

Sign	Derivative	Integral
+ →	$(\ln x)^m$ ↘	x^n
− →	$m(\ln x)^{m-1} \cdot \frac{1}{x}$ →	$\frac{x^{n+1}}{n+1}$

From Table 7, the resulting integral is

$$\int x^n (\ln x)^m dx = \frac{x^{n+1}}{n+1}(\ln x)^m - \frac{m}{n+1}\int (x^n)(\ln x)^{m-1} dx$$

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Remarks: Equation (7) is true for $n \neq$ -1and m is any real number. However, if n = -1 and $m \neq$ -1, the TIBP technique fails, and the integral can be evaluated using the u-substitution which gives

$$\int x^{-1}(\ln x)^{m}\,dx = \frac{1}{m+1}(\ln x)^{m+1} + C \qquad (8)$$

In the same manner, if n = -1 and m = -1, the TIBP fails. Using the u-substitution rule, integration yields

$$\int x^{-1}(\ln x)^{-1}dx = \ln\left|\ln x\right| + C \qquad (9)$$

I. Conclusion and Recommendation

The algorithm of the Tabular Integration by Parts (TIBP) which is based on the algorithm given by Horowitz was modified by forming a table consisting of three columns namely: the S Column where alternating plus and minus signs are written, the D column where f(x) and its successive derivatives are listed, and the I column where g(x) and its successive antiderivatives are listed. The integral is determined by the sum of the successive terms formed by multiplying each entry in the S and D columns by the entry in the I column that lies ***below*** it, and the process may be truncated at any level by forming a remainder term defined as the product of the integral of the entries in S and D columns and the entry in the I column that lies directly ***across*** from it. Some reduction formula that involves the integrals of product of two elementary functions of the forms: $\int x^{n}\sin axdx$, $\int x^{n}\cos axdx$, $\int x^{n}\sinh axdx$, $\int x^{n}\cosh axdx$, $\int x^{n}e^{ax}dx$, and $\int x^{n}(\ln x)^{m}\,dx$, where $n, m, a \in \Re$, were derived using the algorithm of the Tabular Integration by Parts. Further studies are recommended to extend the results of the study by expressing the reduction formula involving the integrals of product of elementary function in series forms.

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