New Efficient Fourth Order Method for Solving Nonlinear Equations

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ABSTRACT

In a paper [Appl. Math. Comput., 188 (2) (2007) 1587--1591], authors have suggested and analyzed a method for solving nonlinear equations. In the present work, we modified this method by using the finite difference scheme, which has a quintic convergence. We have compared this modified Halley method with some other iterative of fifth-orders convergence methods, which shows that this new method having convergence of fourth order, is efficient.

Keywords: Iterative methods, Convergence Order, Halley method, Jarratt method, Numerical Examples

I. INTRODUCTION

In last some years, several iterative type methods have been developed by using the Taylor series, decomposition and quadrature formulae, see [1--13] and the references therein. Using the technique of updating the solution and Taylor series expansion, Noor and Noor [10] have suggested and analyzed a sixth-order predictor--corrector iterative type Halley method for solving the nonlinear equations. Also Kou et al. [6, 7] have also suggested a class of fifth-order iterative methods. In the implementation of these methods, one has to evaluate the second derivative of the function, which is a serious drawback of these methods. To overcome these drawbacks, we modify the predictor--corrector Halley method by replacing the second derivatives of the function f by its finite difference scheme. We prove that the new modified predictor--corrector method is of fifth-order convergence. We also present the comparison of the new method with the methods of Kou et al. [6, 7]. In passing, we would like to point out the results presented by Kou et al. [6, 7] are incorrect. We also rectify this error.

Some examples are given to illustrate the efficiency and robustness of the new proposed method.

II. ITERATIVE METHODS

The Jarratt's fourth-order method [8] which improves the order of convergence is defined by Algorithm 1

$$y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}, \quad f'(x_n) \neq 0 \qquad 1.1$$

$$x_{n+1} = x_n - Jf\,\frac{f(x_n)}{f'(x_n)}, \qquad 1.2$$

where

$$Jf = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}. \qquad 1.3$$

Recently, Kou et al. [5] considered the following two-step iteration scheme

Algorithm 2

$$y_n = x_n - Jf\,\frac{f(x_n)}{f'(x_n)}, \qquad 1.4$$

$$f'(x_n) \neq 0$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, \qquad 1.5$$

where

$$Jf = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}. \qquad 1.6$$

We now state some fifth-order iterative methods which have been suggested by Noor and Noor [9] and Kou et al. [6,7] using quite different techniques.

Algorithm 3

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)} \qquad 1.7$$

$$, \quad f'(x_n) \neq 0$$

$$x_{n+1} = x_n - \frac{2[f(x_n) + f(y_n)]f'(x_n)}{2f'^2(y_n) - [f(x_n) + f(y_n)]f''(x_n)}, \qquad 1.8$$

which is a two-step Halley method of fifth-order convergent.

In a paper Kou et al. [6, 7] have suggested following the iterative methods.

Algorithm 4 *(SHM [7]). For a given x0, compute the approximate solution* x_{n+1} *by the iterative schemes:*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n) f''(x_n)}{2 f'^3(x_n) - 2 f(x_n) f'(x_n) f''(x_n)} \qquad 1.9$$

where $f'(x_n) \neq 0$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) + (y_n - x_n) f''(x_n)}, \qquad 1.10$$

Algorithm 5 *(ISHM [6]). For a given x0, compute the approximate solution* x_{n+1} *by the iterative schemes:*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n) f''(x_n)}{2 f'^3(x_n) - 2 f(x_n) f'(x_n) f''(x_n)} \qquad 1.11$$

where $f'(x_n) \neq 0$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} - \frac{f''(x_n) f(y_n)}{2 f'^3(x_n)}. \qquad 1.12$$

On the basis of above discussion, we propose new Iterative method (FGE):

Algorithm 6: *The iterative technique is given by*

$$y_n = x_n - \frac{2 f(x_n)}{3 f'(x_n)}, \quad f'(x_n) \neq 0 \qquad 3.1$$

$$z_n = x_n - Jf \frac{f(x_n)}{f'(x_n)}, \qquad 3.2$$

$$x_{n+1} = z_n - (1 - Jf) \frac{f(z_n)}{f'(x_n)}, \qquad 3.3$$

where

$$Jf = \frac{3 f'(y_n) + f'(x_n)}{6 f'(y_n) - 2 f'(x_n)}. \qquad 3.4$$

Convergence Analysis of the New Method

In this section, we compute the convergence order of the proposed method (FGE).

Theorem 1 *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is close to α, then the three-step algorithm 6 has fourth order of convergence.*

Proof The iterative technique is given by

$$y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}, \quad f'(x_n) \neq 0 \qquad 3.1$$

$$z_n = x_n - Jf\,\frac{f(x_n)}{f'(x_n)}, \qquad 3.2$$

$$x_{n+1} = z_n - (1 - Jf)\frac{f(z_n)}{f'(x_n)}, \qquad 3.3$$

where

$$Jf = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}. \qquad 3.4$$

Let α be a simple zero of f. By Taylor's expansion, we have,

$$\begin{aligned} f(x_n) = f'(\alpha)[&e_n + c_2 e_n^2 \\ &+ c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 \\ &+ c_7 e_n^7 + c_8 e_n^8 + c_9 e_n^9 \\ &+ c_{10} e_n^{10} + O(e_n^{11})], \end{aligned} \qquad 3.5$$

$$\begin{aligned} f'(x_n) = f'(\alpha)[&1 + 2c_2 e_n \\ &+ 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 \\ &+ 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 \\ &+ 9c_9 e_n^8 + 10c_{10} e_n^9 + O(e_n^{10})], \end{aligned} \qquad 3.6$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, \; k = 2, 3, \ldots, \qquad 3.7$$

and $e_n = x_n - \alpha$.

Using (3.1) , (3.5) and (3.6) we have

$$y_n = \alpha + \frac{1}{3}e_n + \frac{2}{3}c_2 e_n^2$$
$$+(\frac{4}{3}c_3 - \frac{4}{3}c_2^2)e_n^3 + O(e_n^4). \qquad 3.7$$

By Taylor's series, we have

$$f(y_n) = f'(\alpha)[\frac{1}{3}e_n + \frac{7}{9}c_2 e_n^2$$
$$+(\frac{37}{27}c_3 - \frac{8}{9}c_2^2)e_n^3 + O(e_n^4), \qquad 3.8$$

and

$$f'(y_n) = f'(\alpha)[(1 + \frac{2}{3}c_2 e_n$$
$$+(\frac{4}{3}c_2^2 + \frac{1}{3}c_3)e_n^2 + (4c_2 c_3$$
$$-\frac{8}{3}c_2^3 + \frac{4}{27}7c_4)e_n^3 + O(e_n^4). \qquad 3.10$$

Using (3.6) and (3.10) we have

$$Jf = 1 + c_2 e_n + (-c_2^2 + 2c_3)e_n^2$$
$$+(-2c_2 c_3 + \frac{26}{9}c_4)e_n^3 + O(e_n^4). \qquad 3.11$$

Using (3.5), (3.5), (3.6) and (3.11) we have

$$z_n = \alpha + (\frac{1}{9}c_4 + c_2^3 - c_2 c_3)e_n^4$$
$$+(\frac{8}{27}c_5 + 8c_3 c_2^2 - \frac{20}{9}c_2 c_4$$
$$-2c_3^2 - 4c_2^4)e_n^5 + O(e_n^6), \qquad 3.12$$

by Taylor's series, we have

$$f(z_n) = (\frac{1}{9}c_4 + c_2^3 - c_2 c_3)e_n^4$$
$$+(\frac{8}{27}c_5 + 8c_3 c_2^2 - \frac{20}{9}c_2 c_4$$
$$-2c_3^2 - 4c_2^4)e_n^5 + O(e_n^6). \qquad 3.13$$

Using (3.6), (3.6), (3.11) and (3.12) we have

$$x_{n+1} = \alpha + (\frac{1}{9}c_4 - c_2c_3 + c_2^3)e_n^4 + O(e_n^5),$$

implies

$$e_{n+1} = (\frac{1}{9}c_4 - c_2c_3 + c_2^3)e_n^4 + O(e_n^5).$$

Thus we observe that the new three-step method (FGE) has fourth order convergence.

Numerical Examples

In this section we now consider some numerical examples to demonstrate the performance of the newly developed iterative method. We compare classical method (NW), Kou et al. method (see, [6]) (VCM) and (VSHM), Noor et al. methods (see [12]) (NR1) and (NR2) with the new developed method (FGE). All the computations for above mentioned methods, are performed using software Maple 9 , precision 128 digits and $\varepsilon = 10^{-15}$ as tolerance and also the following criteria is used for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \varepsilon,$

(ii) $|f(x_n)| < \varepsilon,$

(iii) Maximum numbers of iterations $= 500$

We used the following examples for comparison:

TABLE OF FUNCTIONS / EXAMPLES	
Functions	Roots
$f_1 = 4x^4 - 4x^2$	1
$f_2 = (x-2)^{23} - 1$	3
$f_3 = \exp(x) \cdot \sin(x) + \ln(x^2+1)$	3.237562984023
$f_4 = (x+2)\exp(x) - 1$	-0.442854401002
$f_5 = x^3 + 4x^2 - 15$	1.631980805566
$f_6 = \exp(x^2 + 7x - 30) - 1$	3
$f_7 = \exp(1-x) - 1$	1
$f_8 = x^3 - 2x^2 - 5$	2.690647448028
$f_9 = (x-1)\exp(-x)$	1
$f_{10} = \frac{1}{x} - 1$	1

Table - 1

	Iterations	$f(x_n)$	δ
$f_1,\ x_0 = 0.75$			
NW	10	7.1e-40	5.9e-21
VCM	33	0	1.9e-42
VSHM	8	-1.0e-127	3.6e-25
NR1	5	1.8e-37	9.5e-20
NR2	11	3.4e-36	4.1e-19
FGE	4	0	3.6e-122
$f_2,\ x_0 = 2.9$			
NW	13	7.0e-44	1.6e-23
VCM	DIVERGE	- - -	- - -
VSHM	DIVERGE	- - - -	- - - -
NR1	6	1.9e-31	8.8e-19
NR2	20	3.1e-29	3.5e-16
FGE	4	0	4.7e-103
$f_3,\ x_0 = 2.9$			
NW	7	-1.1e-51	6.6e-27
VCM	DIVERGE	---	----
VSHM	4	5.0e-127	1.9e-67
NR1	4	-1.0e-9	1.2e-20
NR2	DIVERGE	----	----
FGE	3	5.0e-27	2.5e-69
$f_4,\ x_0 = -.9$			
NW	6	3.4e-29	5.5e-15
VCM	4	1.0e-127	4.8e-26
VSHM	4	-7.0e-128	5.2e-73
NR1	4	3.8e-38	1.8e-19
NR2	45	9.6e-50	2.8e-25
FGE	3	0	1.0e-87

Table - 3

$f_5,\ x_0 = 0.9$			
NW	7	6.1e-51	2.6e-26
VCM	6	1.0e-126	7.7e-43
VSHM	4	0	1.5e-67
NR1	4	5.6e-40	8.0e-21
NR2	14	1.6e-30	4.3e-18
FGE	3	0	3.0e-66
$f_6,\ x_0 = 2.8$			
NW	17	8.2e-33	9.8e-18
VCM	DIVERGE	---	----
VSHM	DIVERGE	5.1e-37	1.0e-18
NR1	8	6.9e-52	2.8e-27
NR2	42	1.9e-33	4.7e-18
FGE	4	0	5.6e-45

Table-4

$f_7, x_0 = 1.1$			
NW	5	7.8e-42	3.9e-21
VCM	3	0	4.3e-39
VSHM	3	0	2.2e-42
NR1	3	2.4e-33	7.0e-17
NR2	4	4.9e-37	9.9e-19
FGE	2	0	9.1e-18
$f_8, x_0 = 2$			
NW	7	1.0e-37	1.3e-19
VCM	53	0	3.7e-29
VSHM	4	-1.0e-126	2.8e-36
NR1	4	7.2e-38	1.0e-19
NR2	9	5.8e-51	3.1e-26
FGE	3	0	5.6e-52

Table-5

f_9, $x_0 = 1$			
NW	1	0	0
VCM	DIVERGE	—	—
VSHM	DIVERGE	—	—
NR1	1	0	0
NR2	DIVERGE	—	—
FGE	1	0	0
f_{10}, $x_0 = 1.5$			
NW	7	2.9e-39	5.4e-20
VCM	4	4.6e-105	3.0e-18
VSHM	4	0	8.3e-41
NR1	3	1.2e-38	1.1e-19
NR2	DIVERGE	—	—
FGE	2	0	0

III. CONCLUSION

In the Table-1 to Table-5, we observe that our iterative method (FGE) is comparable with all the methods cited in the tables, 1-5, and gives better results. With the help of the technique and idea of this paper one can develop higher-order multi-step iterative methods for solving nonlinear equations, as well as a system of nonlinear equations.

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