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THE EXACT BOUNDED SOLUTION TO AN INITIAL BOUNDARY VALUE PROBLEM FOR 1D HYPERBOLIC EQUATION WITH INTERIOR DEGENERACY. I. SEPARATION OF VARIABLES

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Abstract. A 1-parameter initial boundary value problem for the linear homogeneous degenerate wave equation $u_{tt}(t,x;\alpha) - (a(x;\alpha)\, u_x(t,x;\alpha))_x = 0$ (*JODEA*, **28**(1), 1−42) in the space-time rectangle $[0,T]\times[-1,+1]$, where $a(x;\alpha)$ vanishes as $|\bar{x}|^\alpha$ in the subsegment $[-c,+c] \Subset [-1,+1]$, $x = c\bar{x}$, and $\alpha \in (0,2)$, is considered. The IBVP is splitted into three auxiliary IBVPs, involving two undetermined functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$. The auxiliary IBVPs are solved using the method of separation of variables. The matching conditions to gain continuity of the solution $u(t,x;\alpha)$ to the IBVP and its flux are imposed on the solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$ to the auxiliary IBVPs to derive a linear convolution integro-differential system with respect to $h_1(t;\alpha)$ and $h_2(t;\alpha)$.

Key words: degenerate wave equation, separation of variables, linear convolution integro-differential system.

2010 Mathematics Subject Classification: 35L05, 35L35, 35L80.

1. Introduction and the problem formulation

The current study is a sequel to our previous publications [1,2] on the subject dealing with the following 1-parameter initial boundary value problem (IBVP) for the degenerate wave equation in the space-time rectangle $[0,T] \times [-1,+1]$

$$\begin{cases} \dfrac{\partial^2 u(t,x;\alpha)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a(x;\alpha)\,\dfrac{\partial u(t,x;\alpha)}{\partial x}\right), & (t,x) \in (0,T)\times(-1,+1)\,, \\ u(t,-1;\alpha) = h_3(t)\,, & t \in [0,T]\,, \\ u(t,+1;\alpha) = h_0(t)\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [-1,+1]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in (-1,+1)\,, \end{cases} \tag{1.1}$$

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where $h_0, h_3 \in C^2\,([0,T])$ are known control functions, and the 1-parameter x-dependent coefficient function is defined as follows

$$a(x;\alpha) = \begin{cases} a_*|x|^\alpha \equiv |\bar{x}|^\alpha, & 0 \leqslant |x| \leqslant c\,, \\ 1\,, & c \leqslant |x| \leqslant 1\,, \end{cases} \tag{1.2}$$

$\alpha \in (0,2)$, $a_* c^{\,\alpha} = 1$, $x = c\,\bar{x}$, and all the (dependent and independent) variables are non-dimensional. One should refer to [2] to find out more details on the problem formulation.

We also used the following dependent variables

$$p := \frac{\partial u}{\partial t}\,, \qquad q := \frac{\partial u}{\partial x}\,,$$

to simplify notation, and the notion of the flux $f = -aq$, to treat the degenerate wave equation as a conservation law

$$\frac{\partial p}{\partial t} + \frac{\partial f}{\partial x} = 0\,.$$

Here we remind in brief the main results of [2], used in the current study.

I. We found the one-sided (i. e., valid separately for $x<0$ and $x>0$ and marked with the upper indices $\mp$ respectively) power series solutions of the degenerate wave equation using the Frobenius method [6,8]. The only one-sided power series solutions bounded uniformly on α are

$$u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^\theta + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + U^{\mp}_{\alpha,3}(t)\,|x|^{3\theta} + \dots\,, \tag{1.3}$$

where $\theta = \theta(\alpha) = 2 - \alpha$ and the time-dependent coefficient functions obey the recurrence relations

$$U^{\mp\,\prime\prime}_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta + 1]\,a_* U^{\mp}_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,. \tag{1.4}$$

Imposing the constraints $U^{\mp}_{\alpha,\mu}(t) \equiv U_{\alpha,\mu}(t)$ on the above coefficient functions we obtained the following recurrence relations

$$U^{\prime\prime}_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta + 1]\,a_* U_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,, \tag{1.5}$$

and the two-sided series solution

$$u(t,x;\alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^\theta + U_{\alpha,2}(t)\,|x|^{2\theta} + U_{\alpha,3}(t)\,|x|^{3\theta} + \dots \tag{1.6}$$

The flux for the series solution (1.6) was proved to be $f \in C^{(2,1)}\,([0,T] \times [-1,+1])$ and is given as follows

$$f(t,x;\alpha) = -a_*\theta\,x\,\Big(U_{\alpha,1}(t) + 2\,U_{\alpha,2}(t)\,|x|^\theta + 3\,U_{\alpha,3}(t)\,|x|^{2\theta} + \dots\Big)\,. \tag{1.7}$$

A shroud reader could notice that it was sufficient to equate the first two one-sided coefficient functions $U^{\mp}_{\alpha,1}(t)$ and $U^{\mp}_{\alpha,2}(t)$ of the series (1.3) to obtain the following series solution and its flux

$$\begin{aligned} u(t,x;\alpha) &= U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^{\theta} + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + U^{\mp}_{\alpha,3}(t)\,|x|^{3\theta} + \ldots\,, \\ -f(t,x;\alpha) &= a_* \theta\, x\,\Big(U_{\alpha,1}(t) + 2\,U^{\mp}_{\alpha,2}(t)\,|x|^{\theta} + 3\,U^{\mp}_{\alpha,3}(t)\,|x|^{2\theta} + \ldots\Big), \end{aligned}$$

manifesting the set of required properties: 1) $u(\cdot,\cdot;\alpha) \in C^{(2,0)}\left([0,T]\times[-1,+1]\right)$; 2) $f(\cdot,\cdot;\alpha) \in C^{(2,1)}\left([0,T]\times[-1,+1]\right)$. Nevertheless, the recurrence relations (1.4) improved in this way

$$\begin{cases} U''_{\alpha,0}(t) = \theta a_* U_{\alpha,1}(t)\,, \\ U''_{\alpha,1}(t) = 2\theta\,[\theta+1]\,a_* U^{\mp}_{\alpha,2}(t)\,, \\ U^{\mp\,\prime\prime}_{\alpha,\mu}(t) = (\mu+1)\,\theta\,[\mu\theta+1]\,a_* U^{\mp}_{\alpha,\mu+1}(t)\,, \qquad \mu = 2,3,\ldots \end{cases}$$

lead to the two-sided series solution (1.6) again.

II. We used the standard ansatz of the method of separation of variables (SV)

$$u(t,x;\alpha) = O(t;\alpha)\,X(x;\alpha), \qquad (t,x) \in [0,T]\times[-c,+c]\,, \tag{1.8}$$

to find the particular solutions of the degenerate wave equation. The only 2-parameter family of functions $X(x;\alpha)$ that allows particular solutions (1.8) to have the properties of the series solution (1.6) was found to be

$$X(x;\alpha) = |x|^{\frac{\delta}{2}}\,\mathtt{J}_{\varrho}\Big(\lambda\,\Omega\,|x|^{\frac{\theta}{2}}\Big)\,, \qquad |x| \leqslant c\,, \tag{1.9}$$

where $\delta = \delta(\alpha) = 1-\alpha$, $\sqrt{a_*}\,\theta\,\Omega = 2$, λ is a free parameter,

$$\varrho = \varrho(\alpha) = -\frac{\delta}{\theta} = -\frac{1-\alpha}{2-\alpha}\,, \qquad -\frac{1}{2} < \varrho < +\infty\,,$$

$\mathtt{J}_{\varrho}(s)$ is the 1-parameter family of the Bessel functions of the first kind [7, 10] defined as particular solutions to the following second order ordinary differential equation

$$s^2\,\mathtt{J}''_{\varrho}(s) + s\,\mathtt{J}'_{\varrho}(s) = \left(\varrho^2 - s^2\right)\mathtt{J}_{\varrho}(s) \tag{1.10}$$

and having the following power series representation

$$\mathtt{J}_{\varrho}(s) = \Big(\frac{s}{2}\Big)^{\varrho}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}}{\gamma!\,\Gamma(\gamma+\varrho+1)}\Big(\frac{s}{2}\Big)^{2\gamma}. \tag{1.11}$$

Substituting the argument of $\mathtt{J}_{\varrho}$ in (1.9) into the series (1.11) proves that the 2-parameter family (1.9) includes the same power terms $|x|^{\mu\theta}$ as the two-sided series solution (1.6) of the degenerate wave equation (for the details one should refer to the proof of Proposition 2.2 at p. 7). For comparison, we also refer to the recent paper [3], where the Sturm–Liouville problem associated with the degenerate diffusion operator $u \mapsto -(|x|^\alpha u')'$ has been studied in details.

The current study is aimed at obtaining the 1-parameter family of the exact solutions $u(t,x;\alpha)$ to the IBVP (1.1) with continuous and continuously differentiable flux by the method of SV and is arranged as follows.

In Section 2 we give an outline of SV applied to the IBVP. Implementing SV reduces the original IBVP to three auxiliary ones, referred to as IBVP_1, IBVP_2, and IBVP_3. The boundary conditions for the auxiliary problems involve two undetermined functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$ used to match the solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$ to the IBVP_1, IBVP_2, and IBVP_3, and considered as a part of the required solution $u(t,x;\alpha)$.

In Section 3, 4, and 5 we find solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$, and then, in Section 6, to find the required functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$, we impose the matching conditions of Section 2 on the pairs $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$ and $u_2(t,x;\alpha)$, $u_3(t,x;\alpha)$.

Finally, in Sections 7, we treat the resulting matching equations of Sect. 6 as a linear convolution integro-differential system to find the functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$, nevertheless we postpone solving the integro-differential system to our next publication on the subject.

2. Implementing SV to the IBVP

The piecewise representation (1.2) of the coefficient function $a(x;\alpha)$ promts us to replace the IBVP (1.1) posed in the space-time rectangle $[0,T]\times[-1,+1]$ with three auxiliary IBVPs posed in the space-time rectangles: 1) $[0,T]\times[-1,-c]$; 2) $[0,T]\times[-c,+c]$; 3) $[0,T]\times[+c,+1]$, overlapping along the space-time segments $[0,T]\times\{-c\}$ and $[0,T]\times\{+c\}$. The auxiliary problems are referred to as IBVP_1, IBVP_2, and IBVP_3 and posed for the same homogeneous degenerate wave equation. The solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$ to the auxiliary IBVPs satisfy the same zero initial conditions and the following boundary conditions

$$\begin{cases} u_1(t,+c;\alpha) = h_1(t;\alpha)\,, \\ u_1(t,+1;\alpha) = h_0(t)\,, \end{cases} \qquad t \in [0,T]\,, \tag{2.1}$$

$$\begin{cases} u_2(t,-c;\alpha) = h_2(t;\alpha)\,, \\ u_2(t,+c;\alpha) = h_1(t;\alpha)\,, \end{cases} \qquad t \in [0,T]\,, \tag{2.2}$$

$$\begin{cases} u_3(t,-1;\alpha) = h_3(t)\,, \\ u_3(t,-c;\alpha) = h_2(t;\alpha)\,, \end{cases} \qquad t \in [0,T]\,. \tag{2.3}$$

The undetermined functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$ satisfy zero initial conditions

$$h_k(0;\alpha) = 0\,, \qquad h_k'(0;\alpha) = 0\,, \qquad k = 1,2, \tag{2.4}$$

and ensure continuous matching the solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$.

The supplementary conditions imposed on the solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$ are as follows

$$\begin{cases} q_2(t,+c;\alpha) = q_1(t,+c;\alpha)\,, \\ q_3(t,-c;\alpha) = q_2(t,-c;\alpha)\,, \end{cases} \qquad t \in [0,T]\,, \tag{2.5}$$

and ensure continuity of the flux.

The following auxiliary propositions help us to continue the outline of SV applied to the IBVP.

Proposition 2.1. *Let the following Sturm-Liouville problems be given*

$$\begin{cases} X_1''(x) + \lambda_1\, X_1(x) = 0\,, & +c < x < +1\,, \\ X_1(+c) = 0\,, \quad X_1(+1) = 0\,, \end{cases} \tag{2.6}$$

$$\begin{cases} X_3''(x) + \lambda_3\, X_3(x) = 0\,, & -1 < x < -c\,, \\ X_3(-1) = 0\,, \quad X_3(-c) = 0\,, \end{cases} \tag{2.7}$$

then: 1) the eigenvalues and the eigenfunctions of the problems are respectively

$$\lambda_{1,\mu} = \lambda_{3,\mu} = \left(\frac{\mu\pi}{1-c}\right)^2 \equiv \omega_\mu^2\,, \tag{2.8}$$

$$X_{1,\mu}(x) = \mathtt{sin}\left[\omega_\mu(1-x)\right], \tag{2.9}$$

$$X_{3,\mu}(x) = \mathtt{sin}\left[\omega_\mu(1+x)\right], \tag{2.10}$$

2) the eigenfunctions are orthogonal in $L^2(+c,+1)$ and $L^2(-1,-c)$, respectively, that is

$$\int_{+c}^{+1} X_{1,\mu}(x)\, X_{1,\gamma}(x)\,\mathrm{d}x = \begin{cases} 0\,, & \mu \neq \gamma\,, \\ \dfrac{1-c}{2} \equiv \|X_{1,\mu}\|^2_{L^2(+c,+1)}\,, & \mu = \gamma\,, \end{cases}$$

$$\int_{-1}^{-c} X_{3,\mu}(x)\, X_{3,\gamma}(x)\,\mathrm{d}x = \begin{cases} 0\,, & \mu \neq \gamma\,, \\ \dfrac{1-c}{2} \equiv \|X_{3,\mu}\|^2_{L^2(-1,-c)}\,, & \mu = \gamma\,, \end{cases}$$

where $\mu, \gamma \in \mathbb{N}$.

Proposition 2.2. *Let the following Sturm-Liouville problem be given*

$$\begin{cases} \big[a(x;\alpha)X_2'(x;\alpha)\big]' + \lambda_2(\alpha)\, X_2(x;\alpha) = 0\,, & -c < x < +c\,, \\ X_2(-c;\alpha) = X_2(+c;\alpha) = 0\,, \end{cases} \tag{2.11}$$

then: 1) the eigenvalues and the bounded eigenfunctions of the problem are

$$\lambda_{2,\mu}(\alpha) = \left(\frac{\theta}{2}\frac{s_{\varrho,\mu}}{c}\right)^2 \equiv \sigma^2_{\varrho,\mu}\,, \tag{2.12}$$

$$X_{2,\mu}(x;\alpha) = |x|^{\frac{\delta}{2}}\, \mathtt{J}_\varrho\Big(s_{\varrho,\mu}|\bar{x}|^{\frac{\theta}{2}}\Big), \tag{2.13}$$

where $\{s_{\varrho,\mu}\}$ *is the unbounded monotonically increasing sequence of the roots of the equation* $\mathtt{J}_\varrho(s) = 0$, $s > 0$*; 2) the eigenfunctions* (2.13) *are orthogonal in* $L^2(-c,c)$*, that is*

$$\int_{-c}^{+c} X_{2,\mu}(x;\alpha)\, X_{2,\gamma}(x;\alpha)\, \mathrm{d}x = \begin{cases} 0\,, & \mu \neq \gamma\,, \\ \dfrac{2}{\theta}\, c^\theta\, \mathtt{J}^2_{\varrho+1}(s_{\varrho,\mu}) \equiv \|X_{2,\mu}\|^2_{L^2(-c,c)}\,, & \mu = \gamma\,, \end{cases}$$

where $\mu, \gamma \in \mathbb{N}$ *.*

Proof. We start from proving boundedness of the functions (2.13). The Bessel functions of the first kind (1.11) are known [10] to be analytic functions of s except possibly for $s = 0$. Indeed, near $s = 0$ the Bessel functions (1.11) behave as

$$\mathtt{J}_\varrho(s) \sim \frac{1}{\Gamma(\varrho+1)} \left(\frac{s}{2}\right)^\varrho.$$

If $-1 < 2\varrho < 0$ $(0 < \alpha < 1)$, then $\mathtt{J}_\varrho(s)$ are unbounded, whereas if $0 < \varrho < +\infty$ $(1 < \alpha < 2)$, then $\mathtt{J}_\varrho(s)$ are bounded. Representing the functions (2.13) as follows

$$X_{2,\mu}(x;\alpha) = |x|^{\frac{\delta}{2}}\, \mathtt{J}_\varrho(s), \qquad s = s_{\varrho,\mu}|x|^{\frac{\theta}{2}}, \tag{2.14}$$

and accounting for the power series representation (1.11), we obtain for the functions (2.14) the following representation

$$X_{2,\mu}(x;\alpha) = |x|^{\frac{\delta}{2}} \left(\frac{s}{2}\right)^\varrho \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!\,\Gamma(\gamma+\varrho+1)} \left(\frac{s}{2}\right)^{2\gamma} =$$

$$= \left(\frac{s_{\varrho,\mu}}{2}\right)^\varrho \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!\,\Gamma(\gamma+\varrho+1)} \left(\frac{s_{\varrho,\mu}}{2}\right)^{2\gamma} |x|^{\gamma\theta}.$$

It is clear that near $x = 0$ the functions (2.14) behave as

$$X_{2,\mu}(x;\alpha) \sim \frac{1}{\Gamma(\varrho+1)} \left(\frac{s_{\varrho,\mu}}{2}\right)^{\varrho},$$

that is, they are bounded (see Fig. 2.1 and Fig. 2.2 to compare the Bessl functions $\mathtt{J}_0(s_{0,\mu}|\bar{x}|)$ and the eigenfunctions $X_{2,\mu}(x;1) = \mathtt{J}_0\left(s_{0,\mu}|\bar{x}|^{\frac{1}{2}}\right)$, $\mu = 1,5$).

Successive differentiation of the functions (2.14) with respect to x gives

$$\begin{aligned}
X'_{2,\mu}(x;\alpha) &= \mp c^{\frac{\delta}{2}-1}\left[\frac{\delta}{2}\,|\bar{x}|^{\frac{\delta}{2}-1}\,\mathtt{J}_\varrho(s) + \frac{\theta}{2}\,s_{\varrho,\mu}\,|\bar{x}|^{\frac{\delta}{2}+\frac{\theta}{2}-1}\,\mathtt{J}'_\varrho(s)\right],\\
a(x;\alpha)X'_{2,\mu}(x;\alpha) &= \mp c^{\frac{\delta}{2}-1}\left[\frac{\delta}{2}\,|\bar{x}|^{-\frac{\delta}{2}}\,\mathtt{J}_\varrho(s) + \frac{\theta}{2}\,s_{\varrho,\mu}\,|\bar{x}|^{-\frac{\delta}{2}+\frac{\theta}{2}}\,\mathtt{J}'_\varrho(s)\right],\\
\big[a(x;\alpha)X'_{2,\mu}(x;\alpha)\big]' &= c^{\frac{\delta}{2}-2}\,|\bar{x}|^{-\frac{\delta}{2}-1}\left[-\left(\frac{\delta}{2}\right)^2\mathtt{J}_\varrho(s) + \left(\frac{\theta}{2}\right)^2\left(s^2\,\mathtt{J}''_\varrho(s) + s\,\mathtt{J}'_\varrho(s)\right)\right]\\
&= c^{\frac{\delta}{2}-2}\,|\bar{x}|^{\frac{\delta}{2}}\left(|\bar{x}|^{-\frac{\theta}{2}}\right)^2\left[-\left(\frac{\delta}{2}\right)^2 + \left(\frac{\theta}{2}\right)^2\left(\varrho^2 - s^2\right)\right]\mathtt{J}_\varrho(s)\\
&= -\left(\frac{\theta}{2}\frac{s_{\varrho,\mu}}{c}\right)^2 |x|^{\frac{\delta}{2}}\,\mathtt{J}_\varrho\!\left(s_{\varrho,\mu}|\bar{x}|^{\frac{\theta}{2}}\right) = -\sigma^2_{\varrho,\mu}\,X_{2,\mu}(x;\alpha)\,,
\end{aligned}$$

wherefrom we conclude that the functions (2.14) satisfy the differential equation of the problem (2.11). Then, the arguments of the functions (2.14) are augmented in such a way that the boundary conditions of the problem (2.11) are satisfied as well. This completes the proof of the first part of the proposition.

To prove the second part of the proposition, we make use of the change of variables transformation $\xi = \bar{x}^{\frac{\theta}{2}}$

$$\begin{aligned}
\int_{-c}^{+c} X_{2,\mu}(x;\alpha)\,X_{2,\gamma}(x;\alpha)\,\mathrm{d}x &= c^{\theta}\int_{-1}^{+1}|\bar{x}|^{\delta}\,\mathtt{J}_\varrho\!\left(s_{\varrho,\mu}\,|\bar{x}|^{\frac{\theta}{2}}\right)\mathtt{J}_\varrho\!\left(s_{\varrho,\gamma}\,|\bar{x}|^{\frac{\theta}{2}}\right)\mathrm{d}\bar{x}\\
&= 2c^{\theta}\int_{0}^{+1}\bar{x}^{\delta}\,\mathtt{J}_\varrho\!\left(s_{\varrho,\mu}\,\bar{x}^{\frac{\theta}{2}}\right)\mathtt{J}_\varrho\!\left(s_{\varrho,\gamma}\,\bar{x}^{\frac{\theta}{2}}\right)\mathrm{d}\bar{x}\\
&= \frac{4}{\theta}\,c^{\theta}\int_{0}^{1}\xi\,\mathtt{J}_\varrho\big(s_{\varrho,\mu}\,\xi\big)\,\mathtt{J}_\varrho\big(s_{\varrho,\gamma}\,\xi\big)\,\mathrm{d}\xi\,.
\end{aligned}$$

The last integral is known [7, 10] to equal

$$\int_0^1 \xi\,\mathtt{J}_\varrho\big(s_{\varrho,\mu}\,\xi\big)\,\mathtt{J}_\varrho\big(s_{\varrho,\gamma}\,\xi\big)\,\mathrm{d}\xi = \begin{cases} 0\,, & \mu \neq \gamma\,,\\ \dfrac{\mathtt{J}^2_{\varrho+1}(s_{\varrho,\mu})}{2}\,, & \mu = \gamma\,. \end{cases}$$

This completes the proof of the second part of the proposition. □

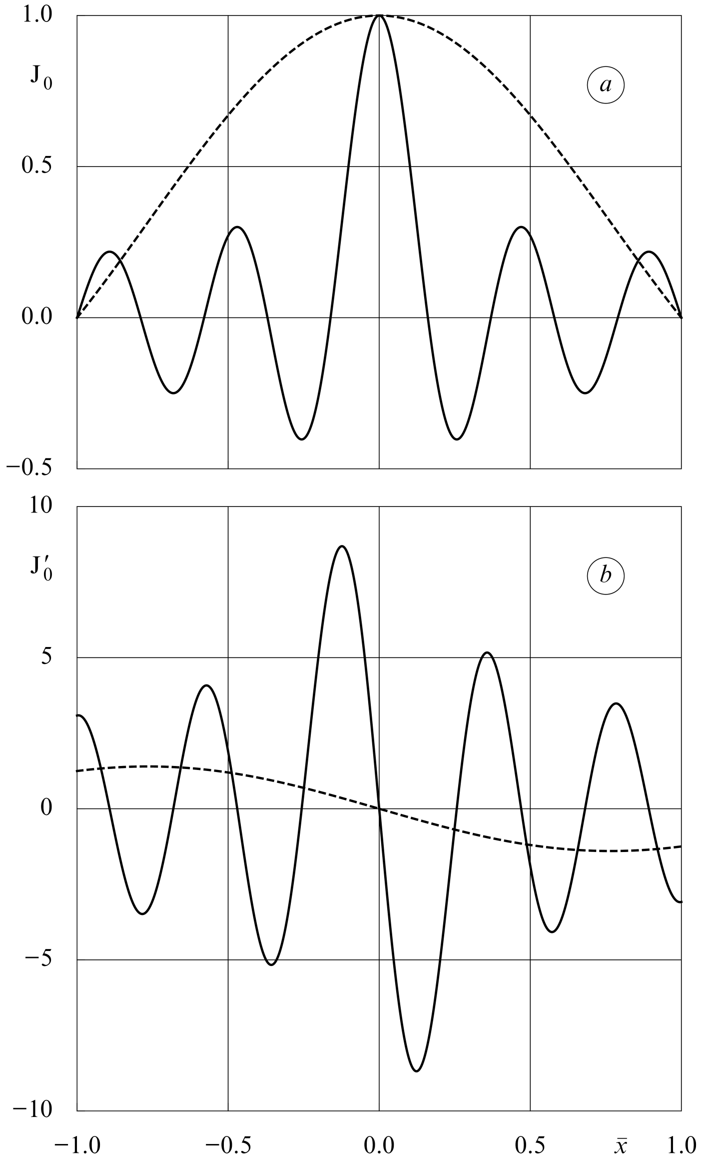

Fig. 2.1. The Bessel functions (1.11) with the augmented argument: $J_0(s_{0,1}\bar{x})$ (dashed) and $J_0(s_{0,5}\bar{x})$ (solid) (a) and $J_0'(s_{0,1}\bar{x})$ (dashed) and $J_0'(s_{0,5}\bar{x})$ (solid) (b)

We continue the outline of SV applied to the IBVP with introducing the following representations for the required solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, $u_3(t,x;\alpha)$

$$\begin{cases} u_1(t,x;\alpha) = v_1(t,x;\alpha) + w_1(t,x;\alpha)\,, \\ u_2(t,x;\alpha) = v_2(t,x;\alpha) + w_2(t,x;\alpha)\,, \\ u_3(t,x;\alpha) = v_3(t,x;\alpha) + w_3(t,x;\alpha)\,, \end{cases} \tag{2.15}$$

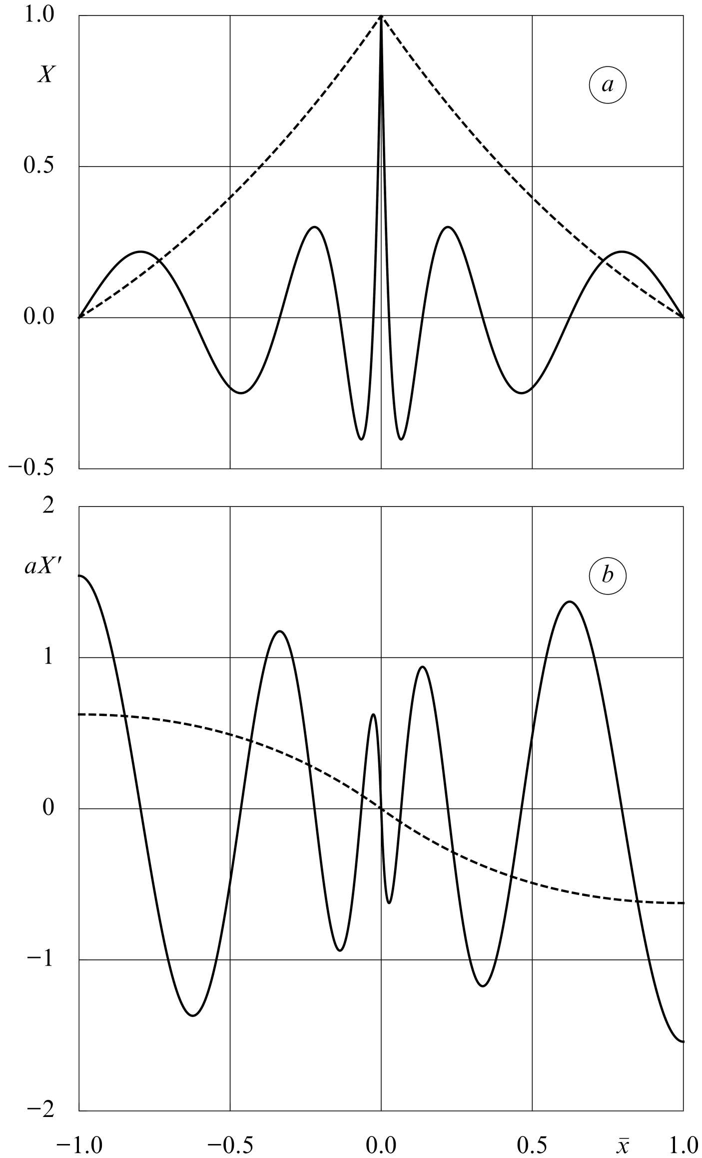

Fig. 2.2. The eigenfunctions (2.13): $X_{2,1}(x;1)$ (dashed) and $X_{2,5}(x;1)$ (solid) (a) and the fluxes $a(x;1)X'_{2,1}(x;1)$ (dashed) and $a(x;1)X'_{2,5}(x;1)$ (solid) (b)

where the functions $v_1(t,x;\alpha)$, $v_2(t,x;\alpha)$, and $v_3(t,x;\alpha)$ are unknown, whereas the functions $w_1(t,x;\alpha)$, $w_2(t,x;\alpha)$, and $w_3(t,x;\alpha)$ are fully determined as follows

$$\begin{cases} w_1(t,x;\alpha) = \phi_{1,1}(x;\alpha)\,h_1(t;\alpha) + \phi_{1,0}(x;\alpha)\,h_0(t)\,, \\ w_2(t,x;\alpha) = \phi_{2,2}(x;\alpha)\,h_2(t;\alpha) + \phi_{2,1}(x;\alpha)\,h_1(t;\alpha)\,, \\ w_3(t,x;\alpha) = \phi_{3,3}(x;\alpha)\,h_3(t) \quad + \phi_{3,2}(x;\alpha)\,h_2(t;\alpha)\,. \end{cases} \tag{2.16}$$

The 'blending' functions

$$\phi_{1,0}(\cdot;\alpha), \phi_{1,1}(\cdot;\alpha) \in C^2([+c,+1])\,,$$
$$\phi_{2,1}(\cdot;\alpha), \phi_{2,2}(\cdot;\alpha) \in C^2([-c,+c])\,,$$
$$\phi_{3,2}(\cdot;\alpha), \phi_{3,3}(\cdot;\alpha) \in C^2([-1,-c])\,,$$

are assumed to satisfy the following 'natural' boundary conditions

$$\begin{cases} \phi_{1,0}(+c;\alpha) = 0\,, & \phi_{1,0}(+1;\alpha) = 1\,, \\ \phi_{1,1}(+c;\alpha) = 1\,, & \phi_{1,1}(+1;\alpha) = 0\,, \end{cases} \tag{2.17}$$

$$\begin{cases} \phi_{2,1}(-c;\alpha) = 0\,, & \phi_{2,1}(+c;\alpha) = 1\,, \\ \phi_{2,2}(-c;\alpha) = 1\,, & \phi_{2,2}(+c;\alpha) = 0\,, \end{cases} \tag{2.18}$$

$$\begin{cases} \phi_{3,2}(-1;\alpha) = 0\,, & \phi_{3,2}(-c;\alpha) = 1\,, \\ \phi_{3,3}(-1;\alpha) = 1\,, & \phi_{3,3}(-c;\alpha) = 0\,. \end{cases} \tag{2.19}$$

The auxiliary IBVP$_1$, IBVP$_2$, and IBVP$_3$ are easily reformulated for the functions $v_1(t,x;\alpha)$, $v_2(t,x;\alpha)$, and $v_3(t,x;\alpha)$ (see Sections 3, 4, 5), and for finding each of these functions we use the standard ansatz of SV

$$\begin{cases} v_1(t,x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t;\alpha)\, X_{1,\mu}(x)\,, \\ v_2(t,x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} O_{2,\mu}(t;\alpha)\, X_{2,\mu}(x;\alpha)\,, \\ v_3(t,x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} O_{3,\mu}(t;\alpha)\, X_{3,\mu}(x)\,, \end{cases} \tag{2.20}$$

where the functions $O_{1,\mu}(t;\alpha)$, $O_{2,\mu}(t;\alpha)$, $O_{3,\mu}(t;\alpha)$ are required.

3. Reformulating and solving IBVP$_1$

The representations (2.15), (2.16) yields to the following reformulation of the auxiliary IBVP$_1$ with respect to $v_1(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 v_1}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial v_1}{\partial x}\right) = g_1\,, & (t,x)\in(0,T)\times(+c,+1)\,, \\ \left.\begin{aligned} v_1(t,+c;\alpha) &= 0 \\ v_1(t,+1;\alpha) &= 0 \end{aligned}\right\}, & t\in[0,T]\,, \\ \left.\begin{aligned} v_1(0,\ x;\alpha) &= 0 \\ \dfrac{\partial v_1(0,\ x;\alpha)}{\partial t} &= 0 \end{aligned}\right\}, & x\in[+c,+1]\,, \end{cases} \tag{3.1}$$

where the right-hand side of the nonhomogeneous degenerate wave equation is expanded into the Fourier series [9] as follows

$$\begin{aligned} g_1(t,x;\alpha) &= -\,\frac{\partial^2 w_1}{\partial t^2} + \frac{\partial}{\partial x}\left(a\,\frac{\partial w_1}{\partial x}\right) \\ &= -\,\phi_{1,1}(x;\alpha)\,h_1''(t;\alpha) - \phi_{1,0}(x;\alpha)\,h_0''(t) \\ &\quad + \phi_{1,1}''(x;\alpha)\,h_1(t;\alpha) + \phi_{1,0}''(x;\alpha)\,h_0(t) \\ &= \sum_{\mu=1}^{\infty} g_{1,\mu}(t;\alpha)\,X_{1,\mu}(x)\,. \end{aligned} \tag{3.2}$$

The coefficients of the above expansion

$$\begin{aligned} g_{1,\mu}(t;\alpha) &= a_{1,\mu}(\alpha)\,h_1''(t;\alpha) + b_{1,\mu}(\alpha)\,h_0''(t) \\ &\quad + \ c_{1,\mu}(\alpha)\,h_1(t;\alpha) + d_{1,\mu}(\alpha)\,h_0\,(t) \end{aligned} \tag{3.3}$$

are determined straightforwardly by integration

$$\begin{cases} a_{1,\mu}(\alpha) = -\dfrac{1}{\|X_{1,\mu}\|^2_{L^2(+c,+1)}} \displaystyle\int_{+c}^{+1} \phi_{1,1}(x;\alpha)\,X_{1,\mu}(x)\,\mathrm{d}x\,, \\ b_{1,\mu}(\alpha) = -\dfrac{1}{\|X_{1,\mu}\|^2_{L^2(+c,+1)}} \displaystyle\int_{+c}^{+1} \phi_{1,0}(x;\alpha)\,X_{1,\mu}(x)\,\mathrm{d}x\,, \\ c_{1,\mu}(\alpha) = +\dfrac{1}{\|X_{1,\mu}\|^2_{L^2(+c,+1)}} \displaystyle\int_{+c}^{+1} \phi_{1,1}''(x;\alpha)\,X_{1,\mu}(x)\,\mathrm{d}x\,, \\ d_{1,\mu}(\alpha) = +\dfrac{1}{\|X_{1,\mu}\|^2_{L^2(+c,+1)}} \displaystyle\int_{+c}^{+1} \phi_{1,0}''(x;\alpha)\,X_{1,\mu}(x)\,\mathrm{d}x\,. \end{cases} \tag{3.4}$$

Then, keeping in mind the ansatz (2.20)

$$v_1(t,x;\alpha) = \sum_{\mu=1}^{\infty} O_{1,\mu}(t;\alpha)\,X_{1,\mu}(x)\,, \tag{3.5}$$

we can easily pose the proper Cauchy problems for finding the functions $O_{1,\mu}(t;\alpha)$

$$\begin{cases} O''_{1,\mu}(t;\alpha) + \omega_\mu^2\, O_{1,\mu}(t;\alpha) = g_{1,\mu}(t;\alpha)\,, \\ O_{1,\mu}(0;\alpha) = 0\,, \quad O'_{1,\mu}(0;\alpha) = 0\,. \end{cases} \tag{3.6}$$

Since the 2-parameter family of particular solutions of the homogeneous ordinary differential equations

$$O''_{1,\mu}(t) + \omega_\mu^2\, O''_{1,\mu}(t) = 0$$

is known to be

$$O_{1,\mu}(t) = A_{1,\mu}\,\mathtt{cos}\left(\omega_\mu t\right) + B_{1,\mu}\,\mathtt{sin}\left(\omega_\mu t\right),$$

where $A_{1,\mu}$ and $B_{1,\mu}$ are undetermined constants (the parameters), we try to find the 2-parameter family of particular solutions of the nonhomogeneous equations of the problems (3.6) following the above solutions as the ansatz

$$O_{1,\mu}(t;\alpha) = A_{1,\mu}(t;\alpha)\,\mathtt{cos}\left(\omega_\mu t\right) + B_{1,\mu}(t;\alpha)\,\mathtt{sin}\left(\omega_\mu t\right),$$

where $A_{1,\mu}$ and $B_{1,\mu}$ are no longer constants but required t-dependent coefficient functions. Substituting the above representation into the ordinary differential equations of the problems (3.6) yields to the systems of linear nonhomogeneous algebraic equations with respect to the first derivatives of the required coefficient functions

$$\begin{cases} +\,\mathtt{cos}\left(\omega_\mu t\right) A'_{1,\mu}(t;\alpha) + \mathtt{sin}\left(\omega_\mu t\right) B'_{1,\mu}(t;\alpha) = 0\,, \\ -\,\mathtt{sin}\left(\omega_\mu t\right) A'_{1,\mu}(t;\alpha) + \mathtt{cos}\left(\omega_\mu t\right) B'_{1,\mu}(t;\alpha) = \omega_\mu^{-1}\, g_{1,\mu}(t;\alpha)\,. \end{cases}$$

The determinants $\Delta_{1,\mu} = \mathtt{cos}^2\left(\omega_\mu t\right) + \mathtt{sin}^2\left(\omega_\mu t\right) \equiv 1$ of the above systems prove the systems to be unconditionally on α solvable and their solutions to read

$$\begin{cases} A'_{1,\mu}(t;\alpha) = -\,\omega_\mu^{-1}\,\mathtt{sin}\left(\omega_\mu t\right)\, g_{1,\mu}(t;\alpha)\,, \\ B'_{1,\mu}(t;\alpha) = +\,\omega_\mu^{-1}\,\mathtt{cos}\left(\omega_\mu t\right)\, g_{1,\mu}(t;\alpha)\,. \end{cases}$$

After integration, we obtain

$$\begin{cases} A_{1,\mu}(t;\alpha) = A^\circ_{1,\mu} - \omega_\mu^{-1} \displaystyle\int_0^t \mathtt{sin}\left(\omega_\mu \tau\right)\, g_{1,\mu}(\tau;\alpha)\,\mathrm{d}\tau\,, \\ B_{1,\mu}(t;\alpha) = B^\circ_{1,\mu} + \omega_\mu^{-1} \displaystyle\int_0^t \mathtt{cos}\left(\omega_\mu \tau\right)\, g_{1,\mu}(\tau;\alpha)\,\mathrm{d}\tau\,, \end{cases}$$

where $A^\circ_{1,\mu}$ and $B^\circ_{1,\mu}$ are undetermined constants. We take zero values of the constants to satisfy the initial conditions of the Cauchy problems (3.6) and to find the required functions of the ansatz (3.5) as follows

$$O_{1,\mu}(t;\alpha) = \omega_\mu^{-1} \int_0^t \Big\{ - \mathtt{sin}\,(\omega_\mu \tau)\, \mathtt{cos}\,(\omega_\mu t) + \mathtt{cos}\,(\omega_\mu \tau)\, \mathtt{sin}\,(\omega_\mu t) \Big\}\, g_{1,\mu}(\tau;\alpha)\, \mathrm{d}\tau$$
$$= \omega_\mu^{-1} \int_0^t \mathtt{sin}\,\big[\omega_\mu (t-\tau)\big]\, g_{1,\mu}(\tau;\alpha)\, \mathrm{d}\tau\,.$$

The above formulas are nothing but the convolutions between trigonometrical sines and the Fourier coefficients (3.3) of the right-hand side of the nonhomogeneous equation of the reformulated IBVP_1 (3.1), therefore, hereinafter, we use the following convenient notation for the solutions of the Cauchy problems (3.6)

$$O_{1,\mu}(t;\alpha) = \omega_\mu^{-1}\, \mathtt{sin}\,(\omega_\mu t) * g_{1,\mu}(t;\alpha)\,, \tag{3.7}$$

then the required solution to the reformulated IBVP_1 (3.1) reads as follows

$$v_1(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1}\, \mathtt{sin}\,(\omega_\mu t) * g_{1,\mu}(t;\alpha)\; X_{1,\mu}(x)\,. \tag{3.8}$$

4. Reformulating and solving IBVP$_2$

The representations (2.15), (2.16) yields to the following reformulation of the auxiliary IBVP_2 with respect to $v_2(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 v_2}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a\, \dfrac{\partial v_2}{\partial x}\right) = g_2\,, & (t,x) \in (0,T)\times(-c,+c)\,, \\ \left.\begin{aligned} v_2(t,-c;\alpha) = 0 \\ v_2(t,+c;\alpha) = 0 \end{aligned}\right\}, & t \in [0,T]\,, \\ \left.\begin{aligned} v_2(0,\ x;\alpha) = 0 \\ \dfrac{\partial v_2(0,\ x;\alpha)}{\partial t} = 0 \end{aligned}\right\}, & x \in [-c,+c]\,, \end{cases} \tag{4.1}$$

where the right-hand side of the nonhomogeneous degenerate wave equation is given in the form of the Fourier – Bessel series expansion [7, 10] as follows

$$\begin{aligned} g_2(t,x;\alpha) &= -\,\frac{\partial^2 w_2}{\partial t^2} + \frac{\partial}{\partial x}\left(a\, \frac{\partial w_2}{\partial x}\right) \\ &= -\,\phi_{2,2}(x;\alpha)\, h_2''(t;\alpha) - \phi_{2,1}(x;\alpha)\, h_1''(t;\alpha) \\ &\quad + \big[a(x;\alpha)\, \phi_{2,2}'(x;\alpha)\big]'\, h_2(t;\alpha) + \big[a(x;\alpha)\, \phi_{2,1}'(x;\alpha)\big]'\, h_1(t;\alpha) \\ &= \sum_{\mu=1}^{\infty} g_{2,\mu}(t;\alpha)\, X_{2,\mu}(x;\alpha)\,. \end{aligned} \tag{4.2}$$

The coefficients of the above expansion

$$\begin{aligned} g_{2,\mu}(t;\alpha) = a_{2,\mu}(\alpha)\, h_2''(t;\alpha) + b_{2,\mu}(\alpha)\, h_1''(t;\alpha) \\ +\ c_{2,\mu}(\alpha)\, h_2(t;\alpha) + d_{2,\mu}(\alpha)\, h_1(t;\alpha) \end{aligned} \tag{4.3}$$

are determined straightforwardly by integration

$$\begin{cases} a_{2,\mu}(\alpha) = -\dfrac{1}{\|X_{2,\mu}\|^2_{L^2(-c,+c)}} \displaystyle\int_{-c}^{+c} \phi_{2,2}(x;\alpha)\, X_{2,\mu}(x;\alpha)\, \mathrm{d}x\,, \\ b_{2,\mu}(\alpha) = -\dfrac{1}{\|X_{2,\mu}\|^2_{L^2(-c,+c)}} \displaystyle\int_{-c}^{+c} \phi_{2,1}(x;\alpha)\, X_{2,\mu}(x;\alpha)\, \mathrm{d}x\,, \\ c_{2,\mu}(\alpha) = +\dfrac{1}{\|X_{2,\mu}\|^2_{L^2(-c,+c)}} \displaystyle\int_{-c}^{+c} \big[a(x;\alpha)\, \phi_{2,2}'(x;\alpha)\big]'\, X_{2,\mu}(x;\alpha)\, \mathrm{d}x\,, \\ d_{2,\mu}(\alpha) = +\dfrac{1}{\|X_{2,\mu}\|^2_{L^2(-c,+c)}} \displaystyle\int_{-c}^{+c} \big[a(x;\alpha)\, \phi_{2,1}'(x;\alpha)\big]'\, X_{2,\mu}(x;\alpha)\, \mathrm{d}x\,. \end{cases} \tag{4.4}$$

The required solution to the reformulated IBVP_2 (4.1) reads immediately as follows

$$v_2(t,x;\alpha) = \sum_{\mu=1}^{\infty} \sigma_{\varrho,\mu}^{-1}\, \mathtt{sin}\left(\sigma_{\varrho,\mu} t\right) *\, g_{2,\mu}(t;\alpha)\ X_{2,\mu}(x;\alpha)\,. \tag{4.5}$$

5. Reformulating and solving $\mathbf{IBVP}_3$

Reformulation of the auxiliary IBVP_3 with respect to $v_3(t,x;\alpha)$ now is clear

$$\begin{cases} \dfrac{\partial^2 v_3}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a\, \dfrac{\partial v_3}{\partial x}\right) = g_3\,, \quad (t,x) \in (0,T)\times(-1,-c)\,, \\ \left.\begin{aligned} v_3(t,-1;\alpha) = 0 \\ v_3(t,-c;\alpha) = 0 \end{aligned}\right\}, \qquad t \in [0,T]\,, \\ \left.\begin{aligned} v_3(0,\ x;\alpha) = 0 \\ \dfrac{\partial v_3(0,\ x;\alpha)}{\partial t} = 0 \end{aligned}\right\}, \qquad x \in [-1,-c]\,. \end{cases} \tag{5.1}$$

The right-hand side of the nonhomogeneous degenerate wave equation is expanded into the Fourier series [9] as follows

$$
\begin{aligned}
g_3(t,x;\alpha) = & -\frac{\partial^2 w_3}{\partial t^2} + \frac{\partial}{\partial x}\left(a\,\frac{\partial w_3}{\partial x}\right) \\
= & -\phi_{3,3}(x;\alpha)\,h_3''(t) - \phi_{3,2}(x;\alpha)\,h_2''(t;\alpha) \\
& + \phi_{3,3}''(x;\alpha)\,h_3(t) + \phi_{3,2}''(x;\alpha)\,h_2(t;\alpha) \\
= & \sum_{\mu=1}^{\infty} g_{3,\mu}(t;\alpha)\,X_{3,\mu}(x)\,,
\end{aligned}
\tag{5.2}
$$

where the coefficients of the expansion

$$
\begin{aligned}
g_{3,\mu}(t;\alpha) = {} & a_{3,\mu}(\alpha)\,h_3''(t) + b_{3,\mu}(\alpha)\,h_2''(t;\alpha) \\
& + c_{3,\mu}(\alpha)\,h_3(t) + d_{3,\mu}(\alpha)\,h_2(t;\alpha)
\end{aligned}
\tag{5.3}
$$

are determined similarly to those of Sect. 3

$$
\begin{cases}
a_{3,\mu}(\alpha) = -\dfrac{1}{\|X_{3,\mu}\|^2_{L^2(-1,-c)}} \displaystyle\int_{-1}^{-c} \phi_{3,3}(x;\alpha)\,X_{3,\mu}(x)\,\mathrm{d}x\,, \\
b_{3,\mu}(\alpha) = -\dfrac{1}{\|X_{3,\mu}\|^2_{L^2(-1,-c)}} \displaystyle\int_{-1}^{-c} \phi_{3,2}(x;\alpha)\,X_{3,\mu}(x)\,\mathrm{d}x\,, \\
c_{3,\mu}(\alpha) = +\dfrac{1}{\|X_{3,\mu}\|^2_{L^2(-1,-c)}} \displaystyle\int_{-1}^{-c} \phi_{3,3}''(x;\alpha)\,X_{3,\mu}(x)\,\mathrm{d}x\,, \\
d_{3,\mu}(\alpha) = +\dfrac{1}{\|X_{3,\mu}\|^2_{L^2(-1,-c)}} \displaystyle\int_{-1}^{-c} \phi_{3,2}''(x;\alpha)\,X_{3,\mu}(x)\,\mathrm{d}x\,.
\end{cases}
\tag{5.4}
$$

The required solution to the reformulated IBVP$_3$ (5.1) reads exactly as (3.8)

$$
v_3(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1}\,\mathtt{sin}\left(\omega_\mu t\right) * g_{3,\mu}(t;\alpha)\;X_{3,\mu}(x)\,. \tag{5.5}
$$

6. Matching the solutions to the IBVPs

In this section we gather the known solutions (3.8), (4.5), (5.5) to the reformulated auxiliary IBVPs (3.1), (4.1), (5.1) and following the representations (2.15), (2.16) of Section 2 obtain the required solutions to the original auxiliary IBVPs

$$\begin{cases} u_1(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1} \texttt{sin}\big(\omega_\mu t\big) * g_{1,\mu}(t;\alpha)\; X_{1,\mu}(x) \\ \qquad\qquad + \phi_{1,1}(x;\alpha)\, h_1(t;\alpha) + \phi_{1,0}(x;\alpha)\, h_0(t)\,, \\ u_2(t,x;\alpha) = \sum_{\mu=1}^{\infty} \sigma_{\varrho,\mu}^{-1} \texttt{sin}\big(\sigma_{\varrho,\mu} t\big) * g_{2,\mu}(t;\alpha)\; X_{2,\mu}(x;\alpha) \\ \qquad\qquad + \phi_{2,2}(x;\alpha)\, h_2(t;\alpha) + \phi_{2,1}(x;\alpha)\, h_1(t;\alpha)\,, \\ u_3(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1} \texttt{sin}\big(\omega_\mu t\big) * g_{3,\mu}(t;\alpha)\; X_{3,\mu}(x) \\ \qquad\qquad + \phi_{3,3}(x;\alpha)\, h_3(t) + \phi_{3,2}(x;\alpha)\, h_2(t;\alpha)\,. \end{cases} \tag{6.1}$$

Then we find the quantities q for the above solutions

$$\begin{cases} q_1(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1} \texttt{sin}\big(\omega_\mu t\big) * g_{1,\mu}(t;\alpha)\; X'_{1,\mu}(x) \\ \qquad\qquad + \phi'_{1,1}(x;\alpha)\, h_1(t;\alpha) + \phi'_{1,0}(x;\alpha)\, h_0(t)\,, \\ q_2(t,x;\alpha) = \sum_{\mu=1}^{\infty} \sigma_{\varrho,\mu}^{-1} \texttt{sin}\big(\sigma_{\varrho,\mu} t\big) * g_{2,\mu}(t;\alpha)\; X'_{2,\mu}(x;\alpha) \\ \qquad\qquad + \phi'_{2,2}(x;\alpha)\, h_2(t;\alpha) + \phi'_{2,1}(x;\alpha)\, h_1(t;\alpha)\,, \\ q_3(t,x;\alpha) = \sum_{\mu=1}^{\infty} \omega_\mu^{-1} \texttt{sin}\big(\omega_\mu t\big) * g_{3,\mu}(t;\alpha)\; X'_{3,\mu}(x) \\ \qquad\qquad + \phi'_{3,3}(x;\alpha)\, h_3(t) + \phi'_{3,2}(x;\alpha)\, h_2(t;\alpha)\,, \end{cases} \tag{6.2}$$

accounting for the derivatives of the eigenfunctions $X_{1,\mu}(x)$ (2.9), $X_{3,\mu}(x)$ (2.10), and $X_{2,\mu}(x;\alpha)$ (2.13) at the points $x = \mp c$ (see the proof of Proposition 2.2)

$$\begin{aligned} X'_{1,\mu}(+c) \quad &= -(-1)^\mu\, \omega_\mu\,, \\ X'_{3,\mu}(-c) \quad &= +(-1)^\mu\, \omega_\mu\,, \\ X'_{2,\mu}(\mp c;\alpha) &= \mp c^{\frac{\delta}{2}}\, \sigma_{\varrho,\mu}\, \mathtt{J}'_\varrho(s_{\varrho,\mu})\,. \end{aligned}$$

Substituting the above values into the expressions for the quantities (6.2) and using the matching conditions (2.5), we obtain the following equalities

$$
\begin{aligned}
q_1(t,+c;\alpha) = & -\sum_{\mu=1}^{\infty}(-1)^{\mu}\,\mathtt{sin}\left(\omega_{\mu}t\right) * g_{1,\mu}(t;\alpha) \\
& + \phi'_{1,1}(+c;\alpha)\,h_1(t;\alpha) + \phi'_{1,0}(+c;\alpha)\,h_0(t) \\
= & +\sum_{\mu=1}^{\infty} c^{\frac{\delta}{2}}\,\mathtt{sin}\left(\sigma_{\varrho,\mu}t\right) * g_{2,\mu}(t;\alpha)\,\mathtt{J}'_{\varrho}(s_{\varrho,\mu}) \\
& + \phi'_{2,2}(+c;\alpha)\,h_2(t;\alpha) + \phi'_{2,1}(+c;\alpha)\,h_1(t;\alpha) = q_2(t,+c;\alpha)\,,
\end{aligned}
$$

$$
\begin{aligned}
q_2(t,-c;\alpha) = & -\sum_{\mu=1}^{\infty} c^{\frac{\delta}{2}}\,\mathtt{sin}\left(\sigma_{\varrho,\mu}t\right) * g_{2,\mu}(t;\alpha)\,\mathtt{J}'_{\varrho}(s_{\varrho,\mu}) \\
& + \phi'_{2,2}(-c;\alpha)\,h_2(t;\alpha) + \phi'_{2,1}(-c;\alpha)\,h_1(t;\alpha) \\
= & +\sum_{\mu=1}^{\infty}(-1)^{\mu}\,\mathtt{sin}\left(\omega_{\mu}t\right) * g_{3,\mu}(t;\alpha) \\
& + \phi'_{3,3}(-c;\alpha)\,h_3(t) + \phi'_{3,2}(-c;\alpha)\,h_2(t;\alpha) = q_3(t,-c;\alpha)\,,
\end{aligned}
$$

to find the functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$.

To make the structure of the above equalities clear, we substitute expressions (3.3), (5.3), and (4.3) for the Fourier coefficients $g_{1,\mu}(t;\alpha)$, $g_{3,\mu}(t;\alpha)$, and for the Fourier – Bessel coefficients $g_{2,\mu}(t;\alpha)$ to obtain

$$
\begin{aligned}
& -\sum_{\mu=1}^{\infty}(-1)^{\mu}\,\mathtt{sin}\left(\omega_{\mu}t\right) * \Big(a_{1,\mu}h''_1 + b_{1,\mu}h''_0 + c_{1,\mu}h_1 + d_{1,\mu}h_0\Big) \\
& + \phi'_{1,1}(+c;\alpha)\,h_1 + \phi'_{1,0}(+c;\alpha)\,h_0 = \phi'_{2,2}(+c;\alpha)\,h_2 + \phi'_{2,1}(+c;\alpha)\,h_1 \qquad (6.3) \\
& +\sum_{\mu=1}^{\infty} c^{\frac{\delta}{2}}\,\mathtt{sin}\left(\sigma_{\varrho,\mu}t\right) * \Big(a_{2,\mu}h''_2 + b_{2,\mu}h''_1 + c_{2,\mu}h_2 + d_{2,\mu}h_1\Big)\,\mathtt{J}'_{\varrho}(s_{\varrho,\mu})\,,
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{\mu=1}^{\infty} c^{\frac{\delta}{2}}\,\mathtt{sin}\left(\sigma_{\varrho,\mu}t\right) * \Big(a_{2,\mu}h''_2 + b_{2,\mu}h''_1 + c_{2,\mu}h_2 + d_{2,\mu}h_1\Big)\,\mathtt{J}'_{\varrho}(s_{\varrho,\mu}) \\
& + \phi'_{2,2}(-c;\alpha)\,h_2 + \phi'_{2,1}(-c;\alpha)\,h_1 = \phi'_{3,3}(-c;\alpha)\,h_3 + \phi'_{3,2}(-c;\alpha)\,h_2 \qquad (6.4) \\
& +\sum_{\mu=1}^{\infty}(-1)^{\mu}\,\mathtt{sin}\left(\omega_{\mu}t\right) * \Big(a_{3,\mu}h''_3 + b_{3,\mu}h''_2 + c_{3,\mu}h_3 + d_{3,\mu}h_2\Big)\,,
\end{aligned}
$$

where the arguments of the functions $h_0(t)$, $h_1(t;\alpha)$, $h_2(t;\alpha)$, $h_3(t;\alpha)$, $h_0''(t)$, $h_1''(t;\alpha)$, $h_2''(t;\alpha)$, and $h_3''(t)$ are not shown, to keep these equalities as simple as possible.

7. Treating the matching equations

The equalities (6.3), (6.4) constitute a linear convolution integro-differential system with respect to the required functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$ of the form

$$\begin{cases} \quad P_{1,0}(\alpha)\,h_1''(t;\alpha) + P_{1,2}(\alpha)\,h_1(t;\alpha) + Q_{1,0}(\alpha)\,h_2''(t;\alpha) + Q_{1,2}(\alpha)\,h_2(t;\alpha) \\ + k_{1,1}(t;\alpha) * \Big(R_{1,0}(\alpha)\,h_1''(t;\alpha) + R_{1,2}(\alpha)\,h_1(t;\alpha)\Big) \\ + k_{1,2}(t;\alpha) * \Big(S_{1,0}(\alpha)\;h_2''(t;\alpha) + S_{1,2}(\alpha)\;h_2(t;\alpha)\Big) = T_1(t;\alpha)\,, \\ \quad P_{2,0}(\alpha)\,h_1''(t;\alpha) + P_{2,2}(\alpha)\,h_1(t;\alpha) + Q_{2,0}(\alpha)\,h_2''(t;\alpha) + Q_{2,2}(\alpha)\,h_2(t;\alpha) \\ + k_{2,1}(t;\alpha) * \Big(R_{2,0}(\alpha)\,h_1''(t;\alpha) + R_{2,2}(\alpha)\,h_1(t;\alpha)\Big) \\ + k_{2,2}(t;\alpha) * \Big(S_{2,0}(\alpha)\;h_2''(t;\alpha) + S_{2,2}(\alpha)\;h_2(t;\alpha)\Big) = T_2(t;\alpha)\,, \end{cases}$$

where $k_{1,1}(t;\alpha)$, $k_{1,2}(t;\alpha)$, $k_{2,1}(t;\alpha)$, and $k_{2,1}(t;\alpha)$ are known kernels.

A very effective tool for solving such systems is known to be the Laplace transformation [4, 5].

8. Conclusions

1. Using the previously obtained results [2] concerning the particular solutions of the degenerate wave equation being under consideration, we succeeded in splitting the original IBVP for the degenerate wave equation into three auxiliary IBVPs involving the undetermined functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$.
2. Using the method of separation of variables, we solved the auxuliary IBVPs.
3. Matching the solutions $u_1(t,x;\alpha)$, $u_2(t,x;\alpha)$, and $u_3(t,x;\alpha)$ to the auxiliary IBVPs, to gain continuity of the required solution $u(t,x;\alpha)$ to the IBVP and its flux $f(t,x;\alpha)$, yields to a linear convolution integro-differential system with respect to the functions $h_1(t;\alpha)$ and $h_2(t;\alpha)$.
4. Solving the obtained system will be published elsewhere.

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