Int. J. Math. And Appl., 8(4)(2020), 75–83
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Some Applications of Chatterjea - Pata Type Fixed Point Theorem in Modular Spaces

Shweta Wasnik[1,*] and Subhashish Biswas[1]

1 Department of Mathematics, Kalinga University, Raipur, Chhattisgarh, India.

Abstract: In this paper, we prove Chatterjea - Pata type fixed point theorem in modular spaces which generalizes and improves some old results. Also we give an application for the existence of solutions of integral equations in modular function spaces.

MSC: 47H10, 34B15, 54H25.

Keywords: Fixed point, Modular spaces, Nonlinear integral equations.

1. Introduction and Preliminaries

Recently, V. Pata [19] improved the Banach principal. Using the idea of Pata, we prove a fixed point theorem in modular spaces. Then we show that how our results generalize old ones. Also, we prepare an application of our main results to the existence of solutions of integral equations in Musielak–Orlicz spaces.

In the first place, we recall some basic notions and facts about modular spaces.

Definition 1.1. *Let X be an arbitrary vector space over $K(=\mathbb{R}$ or $\mathbb{C})$,*

(a). A function $\rho : X \to [0, +\infty]$ is called a modular if

(i). $\rho(x) = 0$ if and only if $x = 0$;

(ii). $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$;

(iii). $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0$, $\beta \geq 0$,

for all $x, y \in X$.

(b). If (iii) is replaced by

(iv). $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0$, $\beta \geq 0$, we say that ρ is convex modular.

(c). A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ given by $X_\rho = \{x \in X : \rho(\lambda x) \to 0$ as $\lambda \to 0\}$.

Example 1.2. *Let $(X, \|.\|)$ be a norm space, then $\|.\|$ is a convex modular on X. But the converse is not true.*

* *E-mail: shwetawasnik511@gmail.com (Research Scholar)*

In general the modular ρ does not behave as a norm or a distance because it is not subadditive. But one can associate to a modular the $F-$norm (see [16]).

Definition 1.3. *The modular space X_ρ can be equipped with the $F-$norm defined by*

$$|x|_\rho = \inf\left\{\alpha > 0;\ \rho\left(\frac{x}{\alpha}\right) \leq \alpha\right\}.$$

Namely, if ρ be convex, then the functional

$$\|x\|_\rho = \inf\left\{\alpha > 0;\ \rho\left(\frac{x}{\alpha}\right) \leq 1\right\},$$

is a norm called the Luxemburg norm in X_ρ which is equivalent to the $F-$norm $|.|_\rho$.

Definition 1.4. *Let X_ρ be a modular space.*

(a). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ρ is said to be:

(i). $\rho-$convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii). $\rho-$Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(b). X_ρ is $\rho-$complete if every $\rho-$Cauchy sequence is $\rho-$convergent.

(c). A subset $B \subseteq X_\rho$ is said to be $\rho-$closed if $\{x_n\}_{n\in\mathbb{N}} \subset B$ with $x_n \to x$, then $x \in B$.

(d). A subset $B \subseteq X_\rho$ is called $\rho-$bounded if $\delta_\rho(B) = \sup\{\rho(x-y) :\ x, y \in B\} < \infty$, where $\delta_\rho(B)$ is called the $\rho-$diameter of B.

(e). We say that ρ has the Fatou property if $\rho(x-y) \leq \liminf \rho(x_n - y_n)$, whenever $\rho(x_n - x) \to 0$, $\rho(y_n - y) \to 0$ as $n \to \infty$.

(f). ρ is said to satisfies the $\triangle_2-$condition if $\rho(x_n) \to 0 \Rightarrow \rho(2x_n) \to 0$ (as $n \to \infty$).

It is easy to check that for every modular ρ and $x, y \in X_\rho$;

(1). $\rho(\alpha x) \leq \rho(\beta x)$ for each $\alpha, \beta \in \mathbb{R}^+$ with $\alpha \leq \beta$.

(2). $\rho(x+y) \leq \rho(2x) + \rho(2y)$.

Now we recall some basic concepts about modular function spaces as formulated by Kozlowski [12].
Let Ω be a nonempty set and Σ be a nontrivial $\sigma-$algebra of subsets of Ω. Let $\mathcal{P}$ be a $\delta-$ring of subsets of Σ such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there is an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. In other words, the family $\mathcal{P}$ plays the role of $\delta-$ring of subsets of finite measure. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$.
By $\mathcal{M}$ we will denote the space of all measurable functions, i.e. all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(w) \to f(w)$ for all $w \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.5. *A function $\rho : \mathcal{E} \times \Sigma \to [0, +\infty]$ is called a function modular if*

(i). $\rho(0, E) = 0$ for any $E \in \Sigma$;

(ii). $\rho(f, E) \leq \rho(g, E)$ whenever $|f(w)| \leq |g(w)|$ for any $w \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$;

(iii). $\rho(f,.) : \Sigma \to [0,+\infty]$ *is a* $\sigma-$*sub-additive measure for every* $f \in \mathcal{E}$*;*

(iv). $\rho(\alpha, A) \to 0$ *as* α *decreases to 0 for every* $A \in \mathcal{P}$*, where* $\rho(\alpha, A) = \rho(\alpha 1_A, A)$*;*

(v). For any $\alpha > 0$*,* $\rho(\alpha, .)$ *is order continuous on* $\mathcal{P}$*, that is* $\rho(\alpha, A_n) \to 0$ *if* $\{A_n\} \in \mathcal{P}$*and decreases to* ϕ*.*

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E);\ g \in \mathcal{E},\ |g(w)| \leq |f(w)|,\ w \in \Omega\}.$$

For simplicity we write $\rho(f)$ instead of $\rho(f, \Omega)$.

One can verify the functional $\rho : \mathcal{M} \to [0, +\infty]$ is a modular in the sense of Definition 1.1. The modular space determined by ρ will be called a modular function space and will be denoted by L_ρ. Recall that

$$L_\rho = \{f \in \mathcal{M} :\ \lim_{\alpha \to 0} \rho(\alpha f) = 0\}.$$

Example 1.6.

(1). The Orlicz modular is defined for every measurable real function f by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|) d\mu(t),$$

where μ *denotes the Lebesgue measure in* $\mathbb{R}$ *and* $\varphi : \mathbb{R} \to [0, \infty)$ *is continuous. We also assume that* $\varphi(u) = 0$ *if and only if* $u = 0$ *and* $\varphi(t) \to \infty$ *as* $t \to \infty$*.*

The modular space induced by the Orlicz modular, is a modular function space and called the Orlicz space.

(2). The Musielak–Orlicz modular spaces (See[14]). Let

$$\rho(f) = \int_{\Omega} \varphi(\omega, |f(\omega)|) d\mu(\omega),$$

where μ *is a* $\sigma-$*finite measure on* Ω *and* $\varphi : \Omega \times \mathbb{R} \to [0, \infty)$ *satisfy the following:*

(i). $\varphi(\omega, u)$ *is a continuous even function of* u *which is non-decreasing for* $u > 0$*; such that* $\varphi(\omega, 0) = 0$*,* $\varphi(\omega, u) > 0$ *for* $u \neq 0$ *and* $\varphi(\omega, u) \to \infty$ *as* $u \to \infty$*.*

(ii). $\varphi(\omega, u)$ *is a measurable function of* ω *for each* $u \in \mathbb{R}$*;*

(iii). $\varphi(\omega, u)$ *is convex function of* u *for each* $\omega \in \Omega$*.*

It is easy to check that ρ *is a convex modular function and the corresponding modular space is called the Musielak–Orlicz spaces and is denoted by* L^φ*.*

In the following we give some notions which will be used in the next sections.

Definition 1.7 (Banach [2])**.** *There exists a number* α*,* $0 \leq \alpha < 1$*, such that, for each* $x, y \in X$*,*

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

In 1972, Chatterjea [4] introduced the following contractive condition:

Definition 1.8 (Chatterjea [4]). *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be Chatterjea contraction if it satisfies the following condition:*

$$d(T(x),T(y)) \leq \frac{k}{2}[d(x,T(y)) + d(y,T(x))]$$

for all $x, y \in X$ and some $k \in [0,1)$.

Remark 1.9. *Note that Banachs contraction and Chatterjeas contraction are independent (see Rhoades paper [20]).*

Definition 1.10 (Khamsi [13]). *Let C be a subset of a modular function space L_ρ. A mapping $T : C \to C$ is called $\rho-$strict contraction if there exists $\lambda < 1$ such that:*

$$\rho(Tf - Tg) \leq \lambda\rho(f - g)$$

for all $f, g \in C$.

Theorem 1.11 (Khamsi [13]). *Let C be a $\rho-$complete, $\rho-$bounded subset of L_ρ and $T : C \to C$ be a $\rho-$strict contraction. Then T has a unique fixed point $z \in C$. Moreover z is the $\rho-$limit of the iterate of any point in C under the action of T.*

Definition 1.12 (Taleb and Hanebaly [21]). *The function $u : I \to L^\varphi$, where $I = [0,A]$ for all $A > 0$, is said to be continuous at $t_0 \in I$ if for $t_n \in I$ and $t_n \to t_0$, then $\rho(u(t_n) - u(t)) \to 0$ as $n \to \infty$.*

If we consider the Musielak–Orlicz modular with $\triangle_2-$condition then the continuity of u at t_0 is equivalent to:

$$(t_n \to t_0) \Rightarrow \|u(t_n) - u(t_0)\|_\rho \to 0 \quad (as \quad n \to \infty).$$

Let $C^\varphi = C(I, L^\varphi)$ be the space of all continuous mappings from $I = [0,A]$ in to L^φ.

Proposition 1.13 (Taleb and Hanebaly [21]). *Suppose that the Musielak–Orlicz modular ρ satisfies $\triangle_2-$condition and $B \subset L^\varphi$ is a $\rho-$closed and convex subset of L^φ. For $a \geq 0$ let $\rho_a(u) = \sup\{e^{-at}\rho(u(t)) : t \in I\}$ for $u \in C^\varphi$, then:*

(1). (C^φ, ρ_a) is a modular space, and ρ_a is a convex modular satisfying the Fatou property and the $\triangle_2-$condition.

(2). C^φ is ρ_a-complete.

(3). $C_0^\varphi = C(I,B)$ is a ρ_a-closed, convex subset of C^φ.

2. Main Results

Let X_ρ be a modular function space, C be a nonempty, $\rho-$complete and $\rho-$bounded subset of X_ρ, x_0 be an arbitrary point in C and $\psi : [0,+\infty) \to [0,+\infty)$ be an increasing function vanishing with continuity at zero. Also consider the vanishing sequence depending on $\alpha \geq 1$, $w_n(\alpha) = \left(\frac{\alpha}{n}\right)^\alpha \sum_{k=1}^{n} \psi\left(\frac{\alpha}{k}\right)$. Let $T : C \to C$ be a mapping, for notational purposes, we define $T^n(x)$, $x \in X_\rho$ and $n \in \{0,1,2,\cdots\}$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$.

Theorem 2.1. *Let $\alpha \geq 1$, $\beta > 0$ and $k \geq 0$ be fixed constants. If the inequality*

$$\rho(Tx - Ty) \leq (1-\epsilon)[\rho(x - Ty) + \rho(y - Tx)] + \epsilon^\alpha\psi(\epsilon)([\rho(x - Ty) + \rho(y - Tx)] + k)^\beta \tag{1}$$

is satisfied for every $\epsilon \in [0,1]$ and every $x, y \in C$, then T has a unique fixed point $z = T(z)$ which is the $\rho-$lim of the iterate of x_0 under the action of T.

Proof. We first show existence. Let $\epsilon = 0$ in (1) thus we get

$$\rho(Tx - Ty) \leq [\rho(x - Ty) + \rho(y - Tx)] \tag{2}$$

for all $x, y \in C$. We construct a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n = T(x_{n-1})$ for all $n \in \mathbb{N}$. Now we claim $\{x_n\}$ is $\rho-$Cauachy sequence in C. By (1), (2) for some $m, n \in \mathbb{N}$ we have

$$\begin{aligned}
\rho(Tx_n - Tx_{n+1}) &\leq (1-\epsilon)[\rho(x_n - Tx_{n+1}) + \rho(x_{n+1} - Tx_n)] + \epsilon^\alpha \psi(\epsilon)(\rho(x_n - Tx_{n+1}) + \rho(x_{n+1} - Tx_n) + k)^\beta \\
\rho(x_{n+1} - x_{n+2}) &\leq (1-\epsilon)[\rho(x_n - x_{n+2}) + \rho(x_{n+1} - x_{n+1})] + \epsilon^\alpha \psi(\epsilon)(\rho(x_n - x_{n+2}) + \rho(x_{n+1} - x_{n+1}) + k)^\beta \\
\rho(x_{n+1} - x_{n+2}) &\leq (1-\epsilon)[\rho(x_n - x_{n+1}) + \rho(x_{n+1} - x_{n+2})] + \epsilon^\alpha \psi(\epsilon)(\rho(x_n - x_{n+1}) + \rho(x_{n+1} - x_{n+2}) + k)^\beta \\
\rho(x_{n+1} - x_{n+2}) &\leq \frac{1-\epsilon}{\epsilon}\rho(x_n - x_{n+1}) + \frac{\epsilon^\alpha \psi(\epsilon)}{\epsilon}(\rho(x_n - x_{n+1}) + \rho(x_{n+1} - x_{n+2}) + k)^\beta \qquad (3)
\end{aligned}$$

Let $M := (2\delta_\rho(C) + k)^\beta$. Since C is $\rho-$bounded, M is finite and from (3) we have

$$\rho(x_{n+1} - x_{n+2}) \leq \frac{1-\epsilon}{\epsilon}\rho(x_n - x_{n+1}) + \epsilon^{\alpha-1}\psi(\epsilon)M.$$

Letting $\epsilon = 1 - \left(\frac{n}{2n+1}\right)^{\alpha-1}$, we have $\epsilon \leq \frac{\alpha-1}{2n+1}$. Keeping in mind that ψ is an increasing function;

$$\begin{aligned}
\rho(x_n - x_{n+1}) \leq \frac{n^{\alpha-1}}{(2n+1)^{\alpha-1}}[\rho(x_{n-1} - x_n) + \rho(x_n - x_{n+1})] + \frac{(\alpha-1)^{\alpha-1}}{(2n+1)^{\alpha-1}}\psi(\frac{\alpha-1}{2n+1})M \\
\Rightarrow [(2n+1)^{\alpha-1} - n^{\alpha-1}]\rho(x_{n+1} - x_n) \leq n^{\alpha-1}\rho(x_n - x_{n-1}) + \alpha^\alpha \psi(\frac{\alpha}{n+1})M. \\
\Rightarrow \rho(x_{n+1} - x_n) \leq \frac{n^{\alpha-1}}{[(2n+1)^{\alpha-1} - n^{\alpha-1}]}\rho(x_n - x_{n-1}) + \frac{(\alpha-1)^{\alpha-1}}{[(2n+1)^{\alpha-1} - n^{\alpha-1}]}\psi(\frac{\alpha}{n+1})M. \qquad (4)
\end{aligned}$$

Letting $r_n := \frac{n^{\alpha-1}}{[(2n+1)^{\alpha-1} - n^{\alpha-1}]}\rho(x_n - x_{n-1})$.

Taking limit as $n \to \infty$ from both sides of (4), we get $\rho(x_{n+1} - x_n) \to 0$ as $n \to \infty$. In general $\rho(x_{n+m} - x_n) \to 0$ as $n \to \infty$. Then $\{x_n\}$ is $\rho-$Cauachy sequence in C. Since C is $\rho-$complete there exists $z \in C$ such that $\rho(x_n - z) \to 0$ as $n \to \infty$. From (1), we get;

$$\begin{aligned}
\rho(\frac{Tz - z}{2}) &\leq \rho(Tz - x_n) + \rho(x_n - z) \\
&\leq (1-\epsilon)\rho(z - x_{n-1}) + \epsilon^\alpha \psi(\epsilon)(\rho(z - x_{n-1}) + k)^\beta + \rho(x_n - z).
\end{aligned}$$

Taking limit as $\epsilon \to 0$ afterwards as $n \to \infty$ we get z is the $\rho-$lim of the iterate of x_0 under the action of T.

To show uniqueness we suppose that y be another fixed point of T. Then from (1), we have

$$\rho(z - y) = \rho(Tz - Ty) \leq (1-\epsilon)2\rho(z - y) + \epsilon^\alpha \psi(\epsilon)(2\rho(z - y) + k)^\beta. \tag{5}$$

Then $\rho(z - y) \leq \epsilon^{\alpha-1}\psi(\epsilon)(2\rho(z - y) + k)^\beta \to 0$ as $\epsilon \to 0$ therefore $z = y$. If for each $\epsilon \in (0, 1]$, strict inequality occurs in (5), then

$$\epsilon^{1-\alpha}\rho(z - y) < \psi(\epsilon)(2\rho(z - y) + k)^\beta.$$

Taking limit as $\epsilon \to 0$, we get contradiction unless $\rho(z - y) = 0$. □

3. Application

In this section, we will study the existence of solution of the following integral equation

$$u(t) = e^{-t} f_0 + \int_o^t e^{s-t} Tu(s) ds, \tag{6}$$

where

(H_1). $T : B \to B$ is ρ-Lipschitz i.e,

$$\exists \ \kappa > 0, \quad \rho(Tu - Tv) \leq \kappa[\rho(u - Tv) + \rho(v - Tu)], \qquad (u, v \in B)$$

(H_2). B is $\rho-$closed, $\rho-$bounded, convex subset of Musielak–Orlicz space L^φ satisfying the $\triangle_2-$condition.

(H_3). $f_0 \in B$ is fixed.

Theorem 3.1. *Under the conditions $H_1 - H_3$, for all $A > 0$, the integral equation (6) has a solution $u \in C^\varphi = C([0, A], L^\varphi)$.*

Proof. Define the operator S on C_0^φ by

$$Su(t) = e^{-t} f_0 + \int_o^t e^{s-t} Tu(s) ds$$

for all $t \in I := [0, A]$.

1st step; first we show that $S : C_0^\varphi \to C_0^\varphi$. Let $u \in C_0^\varphi$ and $t_n, t_0 \in I$ for all $n \in \mathbb{N}$ with $t_n \to t_0$ as $n \to \infty$. We know u is $\rho-$continuous thus $\rho(u(t_n) - u(t_0)) \to 0$. From H_1 we get $\rho(Tu(t_n) - Tu(t_0)) \to 0$ as $n \to \infty$, thus Tu is $\rho-$continuous at t_0. By $\triangle_2-$condition Tu is $\|.\|_\rho-$continuous at t_0, therefore Su is $\|.\|_\rho-$continuous at t_0 consequently is $\rho-$continuous at t_0. Also we have

$$\begin{aligned} \int_0^t e^{s-t} Tu(s) ds &\in \Big(\int_0^t e^{s-t} ds \Big) \overline{co} \{ Tu(s); 0 \leq s \leq t \} \\ &\subseteq (1 - e^{-t}) \overline{co} B, \end{aligned}$$

where $\overline{co}B$ is the closed convex hull of B in $(L^\varphi, \|.\|_\rho)$. But B is convex and $\rho-$closed then $\overline{co}B = B \subseteq \overline{B}_\rho = B$, hence

$$Su(t) \in e^{-t} B + (1 - e^{-t}) B \subseteq B, \quad (\forall t \in I).$$

2st step; we show that C_0^φ is ρ_a-complete and ρ_a-bounded. By Proposition 1.13, C_0^φ is ρ_a-closed subset of ρ_a-complete space C^φ, hence C_0^φ is ρ_a-complete too. Now let $u, v \in C_0^\varphi$. By 1^{st} step $u(t), v(t) \in B$ for all $t \in I$, then

$$\rho_a(u - v) = \sup\{ e^{-at} \rho(u(t) - v(t)); t \in I \} \leq \delta_\rho(B) < \infty,$$

therefore

$$\delta_{\rho_a}(C_0^\varphi) = \sup\{ \rho_a(u - v); u, v \in C_0^\varphi \} < \infty.$$

3st step; for $u, v \in C_0^\varphi$ we have

$$\rho_a(Su - Sv) \leq \kappa \Big(\frac{1 - e^{-(1+a)A}}{1 + a} \Big) [\rho_a(u - Sv) + \rho_a(v - Su)]. \tag{7}$$

Let $w \in C^{\varphi}$ and $\{t_0, t_1, \cdots, t_n\}$ be any division of $[0,t]$. Now suppose

$$\sup\{|t_{i+1} - t_i|, i = 0, 1, \cdots, n-1\} \to 0$$

as $n \to \infty$, then

$$\Big\| \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} w(t_i) - \int_0^t e^{s-t} w(s) ds \Big\|_\rho \to 0.$$

By $\triangle_2-$condition

$$\rho\Big(\sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} w(t_i) - \int_0^t e^{s-t} w(s) ds \Big) \to 0.$$

Using Fatou property, we get

$$\rho\Big(\int_0^t e^{s-t} w(s) ds \Big) \leq \liminf \rho\Big(\sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} w(t_i) \Big). \tag{8}$$

Furthermore

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} \leq \int_0^t e^{s-t} ds \leq 1 - e^{-t} \leq 1 - e^{-A} < 1.$$

By the convexity of ρ, we have

$$\begin{aligned}
\rho\Big(\sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} w(t_i) \Big) &\leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} \rho(w(t_i)) \\
&= \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} e^{a t_i} e^{-a t_i} \rho(w(t_i)) \\
&\leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{(1+a) t_i - t} \rho_a(w) \\
&\leq \Big(\int_0^t e^{(1+a)s - t} ds \Big) \rho_a(w).
\end{aligned}$$

It follows from (8) that

$$\rho\Big(\int_0^t e^{s-t} w(s) ds \Big) \leq \Big(\frac{e^{at} - e^{-t}}{1+a} \Big) \rho_a(w). \tag{9}$$

On the other hand

$$\rho(Su(t) - Sv(t)) = \rho\Big(\int_0^t e^{s-t} (Tu(s) - Tv(s)) \Big) ds.$$

Thus by (9), we have

$$\rho(Su(t) - Sv(t)) \leq \Big(\frac{e^{at} - e^{-t}}{1+a} \Big) \rho_a(Tu - Tv),$$

since T is $\rho-$Lipschitz, we have

$$\begin{aligned}
\rho(Su(t) - Sv(t)) &\leq \Big(\frac{e^{at} - e^{-t}}{1+a} \Big) \sup_{t \in I} e^{-at} \rho(Tu(t) - Tv(t)) \\
&\leq \Big(\frac{e^{at} - e^{-t}}{1+a} \Big) \kappa \sup_{t \in I} e^{-at} [\rho(u(t) - Tv(t)) + \rho(v(t) - Tu(t))] \\
&= \Big(\frac{e^{at} - e^{-t}}{1+a} \Big) \kappa 2 \rho_a(u - v).
\end{aligned}$$

Therefore

$$e^{-at} \rho(Su(t) - Sv(t)) \leq \kappa \Big(\frac{1 - e^{-(1+a)t}}{1+a} \Big) 2 \rho_a(u - v)$$

$$\leq \kappa\Big(\frac{1-e^{-(1+a)A}}{1+a}\Big)2\rho_a(u-v),$$

for all $t \in I$, this implies (7).

4st step; let $\alpha = \beta = 1, \quad k = 0, \quad a > 0$ with;

$$e^{-(1+a)A} > \frac{\kappa-(1+a)}{\kappa}.$$

If we have

$$\frac{\kappa(1-e^{-(1+a)A})}{1+a} \leq (1-\epsilon)+\epsilon^{1+\gamma}K,$$

for all $\gamma > 0, \quad \epsilon \in [0,1]$ and a constant K, then (7) implies that the inequality (1) is satisfied by $\psi(\epsilon) = K\epsilon^{\gamma}$. To this end we define

$$F(\epsilon) = (1-\epsilon)+\epsilon^{1+\gamma}K - \frac{\kappa(1-e^{-(1+a)A})}{1+a}.$$

Now imposing the conditions on F which implies $0 \leq F(\epsilon)$ for all $\epsilon \in [0,1]$, we obtain:

$$K = \frac{\gamma^{\gamma}(1+a)^{\gamma}}{\Big((1+a)(1+\gamma)^{1+\frac{1}{\gamma}} - \kappa(1+\gamma)^{1+\frac{1}{\gamma}}(1-e^{-(1+a)A}\Big)^{\gamma}}.$$

Therefore from steps 1 to 4 and Theorem 2.1, we conclude the existence of fixed point of S which is the solution of integral equation (6). □

Acknowledgement

Authors remain grateful to the honorable reviewers for their kind suggestions for improvement of our paper.

References

[1] M. Abbas, S.H. Khan and A. Razani, *Fixed point theorems of expansive type mappings in modular function spaces*, Fixed Point Theory, 12(2)(2011), 235-240.

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math., 3(1922), 133-181.

[3] B. A. Bin Dehaish and W. M. Kozlowski, *Fixed point iteration processes for asymptotic pointwise nonexpansive mapping in modular function spaces*, Fixed Point Theory Appl., 2012(2012), 118, doi:10.1186/1687-1812-2012-118.

[4] S. K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., 25(1972), 727-730.

[5] S. Dhompongsa, T. Domınguez Benavides, A. Kaewcharoen and B. Panyanak, *Fixed point theorems for multivalued mappings in modular function spaces*, Sci. Math. Jpn., 63(2)(2006), 161-169.

[6] A. P. Farajzadeh, M. Beyg Mohammadi and M. Aslam Noor, *Fixed point theorems in modular spaces*, Math. Commun., 16(2011), 13-20.

[7] N. Hussain, M. A. Khamsi and A. Latif, *Banach operator pairs and common fixed points in modular function spaces*, Fixed Point Theory Appl., 2011(2011), 75, doi:10.1186/1687-1812-2011-75.

[8] M. A. Khamsi, W. M. Kozlowski and S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal., 14(11)(1990), 935-953.

[9] M. A. Kutbi and A. Latif, *Fixed points of multivalued maps in modular function spaces*, Fixed Point Theory Appl., 2009(2009), 786357, doi:10.1155/2009/786357.

[10] M. A. Khamsi, A. Latif and H. Al-Sulami, *KKM and KY fan theorems in modular function spaces*, Fixed Point Theory Appl., 2011(2011), 57, doi:10.1186/1687-1812-2011-57.

[11] W. M. Kozlowski, *Advancements in fixed point theory in modular function spaces*, Arab J. Math., 1(2012), 477–494.

[12] W. M. Kozowski, *Modular Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 122(1988).

[13] M. A. Khamsi, *A convexity property in modular function spaces*, Math. Japonica, 44(2)(1996), 269-279

[14] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math., 18(1959), 49-65.

[15] Ch. Mongkolkeha and P. Kumam, *Some fixed point results for generalized weak contraction mappings in modular spaces*, Int. J. Anal., (2013), 247378, doi:10.1155/2013/247378.

[16] J. Musielak, *Orlicz spaces and Modular spaces*, Lecture Notes in Math, 1034(1983).

[17] S. A. Al-Mezel, A. Al-Roqi and M. A. Khamsi, *One–local retract and common fixed point in modular function spaces*, Fixed Point Theory Appl., 2012(2012), 109, doi:10.1186/1687-1812-2012-109.

[18] H. Nakano, *Modular semi-ordered spaces*, Tokyo Mathematical Book Series, Maruzen Co. Ltd, Tokyo, Japan, (1950).

[19] V. Pata, *A fixed point theorem in metric spaces*, Fixed Point Theory Appl., 10(2011), 299-305.

[20] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226(1977), 257-290.

[21] A. Taleb and E. Hanebaly, *A fixed point theorem and its application to integral equations in modular function spaces*, Proc. Amer. Math. Soc., 128(2)(2000), 419-426.