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A Study on $\alpha-$graceful Labeling of Spade Graph

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Abstract: In this paper we have derived a new graph named spade graph. We have proved two copies of spade graph joined by an arbitrary path is $\alpha-$graceful, n copies of spade graph joined by any vertex is $\alpha-$graceful, path union of n copies of spade graph is $\alpha-$graceful and open star of finite copies of spade graph is also $\alpha-$graceful.

MSC: 05C78.

Keywords: Graceful graph, $\alpha-$graceful graph, path union of a graph G, open star of a graph G, spade graph.

1. Introduction

In 1966 Rosa [1] defined α-labeling as a graceful labeling with an additional property that there is an integer k $(0 \leq k < |E(G)|)$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph which admits α-labeling is necessarily bipartite. A natural generalization of graceful graph is the notion of $k-$graceful graph. Obviously $1-$graceful is graceful. A graph which admits $\alpha-$labeling is always $k-$graceful graph, $\forall\ k \in N$. Ng [3] has identified some graphs that are $k-$graceful, $\forall\ k \in N$ but do not have $\alpha-$labeling.

In [5] Kaneria, Meghpara and Makadia defined open star of graphs and one point union for path of graphs and proved that open star of finite copies of complete bipartite graph, open star of finite copies of grid graph, one point union for path of complete bipartite graphs are graceful graphs. In [6] Kaneria, Teraiya and Meghpara define double path union $D(n \cdot G)$ for a graph G and proved that $D(n \cdot C_m)$, when $m \equiv 0 \pmod 4$, $D(n \cdot K_{r,s})$ and $D(n \cdot P_m)$, when m is even are graceful graphs with $\alpha-$labeling. For a comprehensive bibliography of papers on graph labeling we have refered Gallian [4].

This paper is focused to find $\alpha-$labeling of spade graph and some graph operations on spade graph. We will consider a simple undirected finite graph $G = (V, E)$ on $|V| = p$ vertices and $|E| = q$ edges. For all terminology and standard notations we follows Harary [2]. Here we shall recall some definitions which are used in this paper.

Definition 1.1. *A function f is called* graceful labeling *of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called* graceful graph *if it admits a graceful labeling.*

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Definition 1.2. *A function f is called* $\alpha-$labeling *of a graph $G = (V, E)$ if f is a graceful labeling for G and there exist an integer k $(0 \leq k \leq q-1)$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph G with an $\alpha-$labeling is necessarily bipartite graph. A graph which admits $\alpha-$labeling, we call here* $\alpha-$graceful graph.

Definition 1.3. *Let G be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be n copies of G. The graph obtained by joining vertex v of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall\ i = 1, 2, \ldots, n-1$ is called the* path union *of graph G, It is denoted by $P(n \cdot G)$. If $G = K_1$ then $P(n \cdot K_1) = P_n$.*

Definition 1.4. *A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the connected graphs $G_1, G_2, \ldots, G_n$ is known as* open star of graphs. *We denote such graph by $S(G_1, G_2, \ldots, G_n)$.*

If we replace each vertices of $K_{1,n}$ except the apex vertex by a connected graph G. i.e. $G_1 = G$, $G_2 = G$, $\ldots$, $G_n = G$, such open star of graph is denoted by $S(n \cdot G)$.

Definition 1.5. *A* spade graph *is a planar undirected graph with 5 vertices and 6 edges. It consists of a diamond graph by adding one vertex.*

Illustration 1.6. *Spade graph and its $\alpha-$labeling shown in Figure 1.*

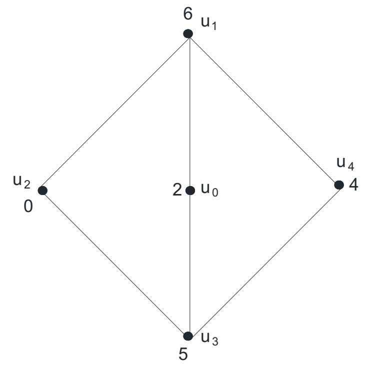

Figure 1. **Spade graph and its $\alpha-$labeling [Here $k = 4$]**

2. Main Results

Theorem 2.1. *The graph G obtained by joining two copies of spade graph by arbitrary path is $\alpha-$graceful.*

Proof. Let G be the graph obtained by joining two copies of spade graph by arbitrary path and g be the spade graph with vertex labeling $u_0 = 2$, $u_1 = 6$, $u_2 = 0$, $u_3 = 5$ and $u_4 = 4$. Let $u_{1,0}, u_{1,1}, \ldots, u_{1,4}, u_{2,0}, u_{2,1}, \ldots, u_{2,4}$ be the vertices of two copies of spade graph and $v_1, v_2, \ldots, v_n$ be the vertices of arbitrary path with $u_{1,3} = v_1$ and $u_{2,3} = v_n$. To define the vertex labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$, consider following two cases, where $q = 12 + (n-1)$ as follows:

Case I: n is even.

$$f(u_{1,j}) = u_j, \qquad \forall\ j = 0, 2, 4;$$

$$f(u_{1,1}) = q;$$

$$f(u_{1,3}) = q - 1 = f(v_1);$$

$$f(v_2) = u_4 + 1;$$

$$
\begin{aligned}
f(v_i) &= f(v_{i-2}) - 1, && \forall \ i = 3, 5, \ldots, n-1;\\
f(v_i) &= f(v_{i-2}) + 1, && \forall \ i = 4, 6, \ldots, n;\\
f(u_{2,3}) &= f(v_n);\\
f(u_{2,1}) &= f(v_n) + 1;\\
f(u_{2,j}) &= f(u_{1,j}) + f(v_n) + 2, && \forall \ j = 0, 2, 4.
\end{aligned}
$$

Above defined labeling pattern give rise f is an injective map and $f^* : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined by $f^*(uv) = | f(u) - f(v) |$, $\forall \ uv \in E(G)$ is bijective, for n is even.

Take $k = \lfloor \frac{q}{2} \rfloor$, $V_1 = \{u_{1,0}, u_{1,2}, u_{1,4}, u_{2,1}, u_{2,3}, v_2, v_4, \ldots, v_n\}$ and $V_2 = \{u_{1,1}, u_{1,3}, u_{2,0}, u_{2,2}, u_{2,4}, v_1, v_3, \ldots, v_{n-1}\}$.

Case II: n is odd.

$$
\begin{aligned}
f(u_{1,j}) &= u_j, && \forall \ j = 0, 2, 4;\\
f(u_{1,1}) &= q;\\
f(u_{1,3}) &= q - 1 = f(v_1);\\
f(v_2) &= u_4 + 1;\\
f(v_i) &= f(v_{i-2}) - 1, && \forall \ i = 3, 5, \ldots, n;\\
f(v_i) &= f(v_{i-2}) + 1, && \forall \ i = 4, 6, \ldots, n-1;\\
f(u_{2,3}) &= f(v_n);\\
f(u_{2,1}) &= f(v_n) - 1;\\
f(u_{2,j}) &= f(u_{1,j}) + f(v_{n-1}) + 1, && \forall \ j = 0, 2, 4.
\end{aligned}
$$

Above defined labeling pattern give rise f is an injective map and $f^* : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined by $f^*(uv) = | f(u) - f(v) |$, $\forall \ uv \in E(G)$ is bijective, for n is odd.

Take $k = \frac{q}{2} + 3$, $V_1 = \{u_{1,0}, u_{1,2}, u_{1,4}, u_{2,0}, u_{2,2}, u_{2,4}, v_2, v_4, \ldots, v_{n-1}\}$ and $V_2 = \{u_{1,1}, u_{1,3}, u_{2,1}, u_{2,3}, v_1, v_3, \ldots, v_n\}$.

Note that G is a bipartite graph, as each $uv \in E(G)$, one end vertex lies in V_1 and another end vertex lies in V_2. Moreover, $f(u) \geq k + 1$, $\forall \ u \in V_2$ and $f(v) \leq k$, $\forall \ v \in V_1$ in G.

Thus, for each $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. Therefore, f is an $\alpha-$graceful labeling for G and so, G is $\alpha-$graceful. □

Theorem 2.2. *The graph G obtained by joining n copies of spade graph is $\alpha-$graceful.*

Proof. Let G be the graph obtained by joining n copies of spade graph. Let $u_{i,0}, u_{i,1}, \ldots, u_{i,4}$; $\forall \ i = 1, 2, \ldots, n$ be the vertices of n copies of spade graph. We shall join $u_{i,4}$ by $u_{i+1,2}$; $\forall \ i = 1, 2, \ldots, n-1$. To define the vertex labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$ as follows, where $p = 5n - (n-1)$ and $q = 6n$.

$$
\begin{aligned}
f(u_{i,0}) &= 4i - 2, && \forall \ i = 1, 2, \ldots, n;\\
f(u_{i,2}) &= 4(i - 1), && \forall \ i = 1, 2, \ldots, n;\\
f(u_{i,4}) &= 4i, && \forall \ i = 1, 2, \ldots, n;\\
f(u_{1,1}) &= q;\\
f(u_{1,3}) &= q - 1;
\end{aligned}
$$

$$f(u_{i,1}) = f(u_{i-1,1}) - 2, \qquad \forall\ i = 2, 3, \ldots, n;$$
$$f(u_{i,3}) = f(u_{i-1,3}) - 2, \qquad \forall\ i = 2, 3, \ldots, n.$$

Above defined labeling pattern give rise f is an injective map and $f^* : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined by $f^*(uv) = \mid f(u) - f(v) \mid, \forall\ uv \in E(G)$ is bijective.

Take $k = \frac{2q}{3}$, $V_1 = \{u_{i,0}, u_{i,2}, u_{i,4}\}$ and $V_2 = \{u_{i,1}, u_{i,3}\}, \forall\ i = 1, 2, \ldots, n$.

Note that G is a bipartite graph, as each $uv \in E(G)$, one end vertex lies in V_1 and another end vertex lies in V_2. Moreover, $f(u) \geq k+1$, $\forall\ u \in V_2$ and $f(v) \leq k$, $\forall\ v \in V_1$ in G.

Thus, for each $uv \in E(G)$, $min\{f(u), f(v)\} \leq k < max\{f(u), f(v)\}$. Therefore, f is an $\alpha-$graceful labeling for G and so, G is $\alpha-$graceful. □

Theorem 2.3. *The path union of n copies of spade graph is $\alpha-$graceful.*

Proof. Let $G = (V, E)$ be the path union of n copies of spade graph obtained by joining any vertex of one cycle to any vertex of next cycle and g be the spade graph with vertex labeling $u_0 = 2$, $u_1 = 6$, $u_2 = 0$, $u_3 = 5$, $u_4 = 4$, where $p = 5$, $q = 6$. Let $u_{i,j}$; $\forall\ i = 1, 2, \ldots, n$, $\forall\ j = 0, 1, \ldots, 4$ be the vertices of n copies of spade graph. we shall join $u_{i,j}$ with $u_{i+1,j}$; $\forall\ i = 1, 2, \ldots, n-1$, $\forall\ j = 0, 1, \ldots, 4$ by an edge. G has $P = 5n$ vertices and $Q = 6n + (n-1)$ edges.

Define $f : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows:

$$\begin{aligned}
f(u_{1,j}) &= u_j, && \text{for } j = 0, 2, 4;\\
f(u_{1,j}) &= u_j + (Q - q), && \text{for } j = 1, 3;\\
f(u_{2,j}) &= f(u_{1,j}) + (Q - q), \quad f(u_{1,j}) < \frac{Q}{2}, &&\\
&= f(u_{1,j}) - (Q - q), \quad f(u_{1,j}) > \frac{Q}{2}, && \forall\ j = 0, 1, \ldots, 4;\\
f(u_{i,j}) &= f(u_{i-2,j}) - (q + 1), \quad f(u_{i-2,j}) > \frac{Q}{2}, &&\\
&= f(u_{i-2,j}) + (q + 1), \quad f(u_{i-2,j}) < \frac{Q}{2}, && \forall\ i = 3, 4, \ldots, n, \quad \forall\ j = 0, 1, \ldots, 4.
\end{aligned}$$

Above defined labeling pattern give rise f is an injective map and $f^* : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined by $f^*(uv) = \mid f(u) - f(v) \mid, \forall\ uv \in E(G)$ is bijective, for n is even. Take $k = \frac{Q+2}{2}$, where n is odd,
$V_1 = \{u_{i,j}/i = 1, 3, \ldots, n,\ \ j = 0, 2, 4\} \cup \{u_{i,j}/i = 2, 4, \ldots, n-1,\ \ j = 1, 3\}$ and $V_2 = \{u_{i,j}/i = 1, 3, \ldots, n,\ \ j = 1, 3\} \cup \{u_{i,j}/i = 2, 4, \ldots, n-1,\ \ j = 0, 2, 4\}$ and $k = \frac{Q-1}{2}$, where n is even, $V_1 = \{u_{i,j}/i = 1, 3, \ldots, n-1,\ \ j = 0, 2, 4\} \cup \{u_{i,j}/i = 2, 4, \ldots, n,\ \ j = 1, 3\}$ and $V_2 = \{u_{i,j}/i = 1, 3, \ldots, n-1, j = 1, 3\} \cup \{u_{i,j}/i = 2, 4, \ldots, n, j = 0, 2, 4\}$.

Note that G is a bipartite graph, as each $uv \in E(G)$, one end vertex lies in V_1 and another end vertex lies in V_2. Moreover, $f(u) \geq k+1$, $\forall\ u \in V_2$ and $f(v) \leq k$, $\forall\ v \in V_1$ in G.

Thus, for each $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. Therefore, f is an $\alpha-$graceful labeling for G and so, path union of n copies of spade graph is $\alpha-$graceful. □

Theorem 2.4. *Open star of finite copies of spade graph is $\alpha-$graceful.*

Proof. Let G be the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the spade graph. Let u_0 is the central vertex for the graph G. i.e. it is apex vertex of the original graph $K_{1,t}$. Let $u_{i,j} (1 \leq i \leq t, 0 \leq j \leq 4)$ be the vertices of i^{th} copy of spade graph in G.

We shall join $u_{i,1}$ with the vertex u_0 by an edge to form the open star of graphs G; $\forall\ i = 1, 2, \ldots, t$.

We know that the spade graph is a graceful graph on $p = 5$ vertices and $q = 6$ edges. Let $u_0 = 2$, $u_1 = 6$, $u_2 = 0$, $u_3 = 5$ and $u_4 = 4$ be the vertex labeling of spade graph.

We define vertex labeling function $f : V(G) \longrightarrow \{0, 1, \dots, Q\}$, where $P = 5t + 1$ vertices and $Q = 7t$ edges as follows:

$$
\begin{aligned}
f(u_0) &= 0; \\
f(u_{1,j}) &= u_j + 1, && \text{for } j = 0, 2, 4, \\
&= u_j + (Q - q), && \text{for } j = 1, 3; \\
f(u_{2,j}) &= f(u_{1,j}) + (Q - (q + 1)), && \text{when } f(u_{1,j}) < \frac{Q}{2}, \\
&= f(u_{1,j}) - (Q - (q + 1)), && \text{when } f(u_{1,j}) > \frac{Q}{2}, \ \forall\ j = 0, 1, \dots, 4; \\
f(u_{i,j}) &= f(u_{i-2,j}) + (q + 1), && \text{when } f(u_{i-2,j}) < \frac{Q}{2}, \\
&= f(u_{i-2,j}) - (q + 1), && \text{when } f(u_{i-2,j}) > \frac{Q}{2}, \ \forall\ i = 3, 4, \dots, t, \ \forall\ j = 0, 1, \dots, 4.
\end{aligned}
$$

Above defined labeling pattern give rise f is an injective map and $f^* : E(G) \longrightarrow \{1, 2, \dots, q\}$ defined by $f^*(uv) = \mid f(u) - f(v) \mid$, $\forall\ uv \in E(G)$ is bijective, for n is even. Take $k = \frac{Q}{2}$, where t is even,
$V_1 = \{u_{i,j}/i = 1, 3, \dots, t-1, j = 0, 2, 4\} \cup \{u_{i,j}/i = 2, 4, \dots, t, j = 1, 3\}$ and $V_2 = \{u_{i,j}/i = 1, 3, \dots, t-1, j = 1, 3\} \cup \{u_{i,j}/i = 2, 4, \dots, t, j = 0, 2, 4\}$ and $k = \frac{Q+3}{2}$, where t is odd, $V_1 = \{u_{i,j}/i = 1, 3, \dots, t, j = 0, 2, 4\} \cup \{u_{i,j}/i = 2, 4, \dots, t-1, j = 1, 3\}$ and $V_2 = \{u_{i,j}/i = 1, 3, \dots, t, j = 1, 3\} \cup \{u_{i,j}/i = 2, 4, \dots, t-1, j = 0, 2, 4\}$.

Note that G is a bipartite graph, as each $uv \in E(G)$, one end vertex lies in V_1 and another end vertex lies in V_2. Moreover, $f(u) \geq k + 1$, $\forall\ u \in V_2$ and $f(v) \leq k$, $\forall\ v \in V_1$ in G.

Thus, for each $uv \in E(G)$, $min\{f(u), f(v)\} \leq k < max\{f(u), f(v)\}$. Therefore, f is an $\alpha-$graceful labeling for G and so, open star of t copies of spade graph is $\alpha-$graceful. $\square$

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